Math 314/814, Section 6

Quiz number 11 Solution

Find the orthogonal projection $\text{proj}_W(\vec{v})$ of the vector $\vec{v} = \begin{pmatrix} 1 \\ 2 \\ 3 \end{pmatrix}$ onto the subspace W spanned by the vectors $\begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix}$ and $\begin{pmatrix} 2 \\ -1 \\ 0 \end{pmatrix}$.

$W = \text{Col}(A)$ for $A = \begin{pmatrix} 1 & 2 \\ 0 & -1 \\ 1 & 0 \end{pmatrix}$, and to find the vector $\vec{w} = A\vec{x}$ in $\text{Col}(A)$ with $\vec{v} - A\vec{x} \in W^\perp = \text{Null}(A^T)$ we need $A^T(\vec{v} - A\vec{x}) = \vec{0}$,
i.e., we solve $(A^TA)\vec{x} = A^T\vec{v}$ and then set $\vec{w} = A\vec{x}$.

We have $A^TA = \begin{pmatrix} 1 & 0 & 1 \\ 2 & -1 & 0 \end{pmatrix} \begin{pmatrix} 1 & 2 \\ 0 & -1 \\ 1 & 0 \end{pmatrix} = \begin{pmatrix} 2 & 2 \\ 2 & 5 \end{pmatrix}$, and $A^T\vec{v} = \begin{pmatrix} 1 & 0 & 1 \\ 2 & -1 & 0 \end{pmatrix} \begin{pmatrix} 1 \\ 2 \\ 3 \end{pmatrix} = \begin{pmatrix} 4 \\ 0 \end{pmatrix}$,
so we solve our system by row reduction:

$$\left(\begin{array}{cc|c} 2 & 2 & 4 \\ 2 & 5 & 0 \end{array}\right) \to \left(\begin{array}{cc|c} 1 & 1 & 2 \\ 2 & 5 & 0 \end{array}\right) \to \left(\begin{array}{cc|c} 1 & 1 & 2 \\ 0 & 3 & -4 \end{array}\right) \to \left(\begin{array}{cc|c} 1 & 1 & 2 \\ 0 & 1 & -4/3 \end{array}\right) \to \left(\begin{array}{cc|c} 1 & 0 & 10/3 \\ 0 & 1 & -4/3 \end{array}\right)$$

so $\vec{x} = \begin{pmatrix} 10/3 \\ -4/3 \end{pmatrix}$. Then $\vec{w} = A\vec{x} = \begin{pmatrix} 1 & 2 \\ 0 & -1 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} 10/3 \\ -4/3 \end{pmatrix} = \begin{pmatrix} 2/3 \\ 4/3 \\ 10/3 \end{pmatrix}$, so $\text{proj}_W(\vec{v}) = \begin{pmatrix} 2/3 \\ 4/3 \\ 10/3 \end{pmatrix}$.

Alternatively, we can, since the columns of A are linearly independent, use the formula $\text{proj}_W(\vec{v}) = A(A^TA)^{-1}A^T\vec{v}$.

Since $A^TA = \begin{pmatrix} 2 & 2 \\ 2 & 5 \end{pmatrix}$ has determinant $\det(A) = 10 - 4 = 6$, we have

$(A^TA)^{-1} = \begin{pmatrix} 5/6 & -2/6 \\ -2/6 & 2/6 \end{pmatrix} = \begin{pmatrix} 5/6 & -1/3 \\ -1/3 & 1/3 \end{pmatrix}$, so

$A^T\vec{v} = \begin{pmatrix} 1 & 0 & 1 \\ 2 & -1 & 0 \end{pmatrix} \begin{pmatrix} 1 \\ 2 \\ 3 \end{pmatrix} = \begin{pmatrix} 4 \\ 0 \end{pmatrix}$, and

$(A^TA)^{-1}A^T\vec{v} = \begin{pmatrix} 5/6 & -1/3 \\ -1/3 & 1/3 \end{pmatrix} \begin{pmatrix} 4 \\ 0 \end{pmatrix} = \begin{pmatrix} 4(5/6) \\ 4(-1/3) \end{pmatrix} = \begin{pmatrix} 10/3 \\ -4/3 \end{pmatrix}$, and so

$\text{proj}_W(\vec{v}) = \begin{pmatrix} 1 & 2 \\ 0 & -1 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} 10/3 \\ -4/3 \end{pmatrix} = \begin{pmatrix} 2/3 \\ 4/3 \\ 10/3 \end{pmatrix}$.