Math 314/814, Section 5

Quiz number 6 Solution

Use row reduction to find the determinant of the matrix

$$A = \begin{pmatrix} 3 & 1 & 6 \\ 2 & -1 & 5 \\ 1 & 2 & 4 \end{pmatrix} .$$

If we keep track of which row operations we use as we row reduce A, we can compute the determinant:

$$A = \begin{pmatrix} 3 & 1 & 6 \\ 2 & -1 & 5 \\ 1 & 2 & 4 \end{pmatrix} \xrightarrow{E_{13}} \begin{pmatrix} 1 & 2 & 4 \\ 2 & -1 & 5 \\ 3 & 1 & 6 \end{pmatrix} \xrightarrow{E_{12}(-2)} \begin{pmatrix} 1 & 2 & 4 \\ 0 & -5 & -3 \\ 3 & 1 & 6 \end{pmatrix} \xrightarrow{E_{13}(-3)} \begin{pmatrix} 1 & 2 & 4 \\ 0 & -5 & -3 \\ 0 & -5 & -6 \end{pmatrix}$$

$$\xrightarrow{E_2(-\frac{1}{5})} \begin{pmatrix} 1 & 2 & 4 \\ 0 & 1 & 3/5 \\ 0 & -5 & -6 \end{pmatrix} \xrightarrow{E_{23}(5)} \begin{pmatrix} 1 & 2 & 4 \\ 0 & 1 & 3/5 \\ 0 & 0 & -3 \end{pmatrix} \xrightarrow{E_3(-\frac{1}{3})} \begin{pmatrix} 1 & 2 & 4 \\ 0 & 1 & 3/5 \\ 0 & 0 & 1 \end{pmatrix}$$

$$\xrightarrow{E_{32}(-\frac{3}{5})} \begin{pmatrix} 1 & 2 & 4 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \xrightarrow{E_{31}(-4)} \begin{pmatrix} 1 & 2 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \xrightarrow{E_{21}(-2)} \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} = I_3 \text{ , so}$$

$I_3 = E_{21}(-2)E_{31}(-4)E_{32}(-\frac{3}{5})E_3(-\frac{1}{3})E_{23}(5)E_2(-\frac{1}{5})E_{13}(-3)E_{12}(-2)E_{13}A$, and so

$1 = \det(I_3) = (1)(1)(1)(-\frac{1}{3})(1)(-\frac{1}{5})(1)(1)(-1)\det(A) = -\frac{1}{15}\det(A)$, and so

$\det(A) = -15$.

Alternatively,

$$A = \begin{pmatrix} 3 & 1 & 6 \\ 2 & -1 & 5 \\ 1 & 2 & 4 \end{pmatrix} \xrightarrow{E_{13}} \begin{pmatrix} 1 & 2 & 4 \\ 2 & -1 & 5 \\ 3 & 1 & 6 \end{pmatrix} \xrightarrow{E_{12}(-2)} \begin{pmatrix} 1 & 2 & 4 \\ 0 & -5 & -3 \\ 3 & 1 & 6 \end{pmatrix}$$

$$\xrightarrow{E_{13}(-3)} \begin{pmatrix} 1 & 2 & 4 \\ 0 & -5 & -3 \\ 0 & -5 & -6 \end{pmatrix} \xrightarrow{E_{23}(-1)} \begin{pmatrix} 1 & 2 & 4 \\ 0 & -5 & -3 \\ 0 & 0 & -3 \end{pmatrix} = R$$

which ends at an upper triangular matrix $R = \begin{pmatrix} 1 & 2 & 4 \\ 0 & -5 & -3 \\ 0 & 0 & -3 \end{pmatrix}$,

with determinant $(1)(-5)(-3) = 15$, and so

$15 = \det(R) = \det[E_{23}(-1)E_{13}(-3)E_{12}(-2)E_{13}A] = (1)(1)(1)(-1)\det(A) = -\det(A)$,

so $\det(A) = -15$.