Math 314

Topics for first exam

Chapter 1: Linear systems of equations

§1: **Some examples**

Systems of linear equations:

$$2x - 3y - z = 6$$
$$3x + 2y + z = 7$$

Goal: find simultaneous solutions: all x, y, z satisfying both equations.

Most general type of system:

$$a_{11}x_1 + \cdots + a_{1n}x_n = b_1$$
$$\cdots$$
$$a_{m1}x_1 + \cdots + a_{mn}x_n = b_m$$

Example: input-output models

§2: **Notations and a review of numbers**

Set notation: $A \cup B$, $A \cap B$, $A \setminus B$

Number systems: natural, integers, rational, reals, complex

Some complex arithmetic:

$i = \sqrt{-1}$, pretend i behaves like a real number

complex numbers: standard form $z = a + bi$; addition, subtraction, multiplication

division: complex conjugate $\overline{z} = a - bi$

$\overline{z+w} = \overline{z} + \overline{w}$; $\overline{zw} = \overline{z}\,\overline{w}$

$z \cdot \overline{z} = a^2 + b^2$ (<u>real</u>!) ; $z_1/z_2 = (z_1 \cdot \overline{z_2})/(z_2 \cdot \overline{z_2})$

Polar coordinates:

$z = a + bi$ (complex number) $= (a, b)$ (point in plane) $=$

(r, θ) (distance from origin and angle with (positive) x-axis)

$z = a + bi = r(\cos\theta + i\sin\theta) = re^{i\theta}$, $w = c + di = s(\cos\phi + i\sin\phi) = se^{i\phi}$, then

$zw = rs(\cos(\theta+\phi) + i\sin(\theta+\phi) = (rs)e^{i(\theta+\phi)}$.   setting $z = w$ yields

$z^n = r^n e^{i(n\theta)}$ (DeMoivre's formula)

Think backwards; solve $z^n = w$

Need: $r^n = s$, $\cos(n\theta) = \cos\phi$, $\sin(n\theta) = \sin\phi$; i.e.

$r = s^{1/n}$, $n\theta = \phi + 2k\pi$, i.e., $\theta = \phi/n + 2k\pi/n$

So $z^n = w$ has n distinct solutions, coming from $k = 0, 1, \ldots, n-1$

§3: **Gaussian elimination: basic ideas**

$$3x + 5y = 2$$
$$2x + 3y = 1$$

Idea use $3x$ in first equation to eliminate $2x$ in second equation. How? Add a multiply of first equation to second. Then use y-term in new second equation to remove $5y$ from first! The point: a solution to the original equations **must** also solve the new equations. The **real** point: it's much easier to figure out the solutions of the new equations!

Streamlining: keep only the essential information; throw away unneeded symbols!

3x+5y=2 replace
2x+3y=1 with

$$\left(\begin{array}{cc|c} 3 & 5 & 2 \\ 2 & 3 & 1 \end{array}\right)$$

We get an **(augmented) matrix** representing the system of equations. We carry out the same operations we used with equations, but do them to the rows of the matrix.
Three basic operations (elementary row operations):
E_{ij} : switch ith and jth rows of the matrix
$E_{ij}(m)$: add m times jth row to the ith row
$E_i(m)$: multiply ith row by m

Terminology: first non-zero entry of a row = **leading entry**; leading entry used to zero out a column = **pivot**.
Basic procedure (Gauss-Jordan elimination): find non-zero entry in first column, switch up to first row (E_{1j}) (pivot in (1,1) position). Use $E_1(m)$ to make first entry a 1, then use $E_{1j}(m)$ operations to zero out the other entries of the first column. Then: find leftmost entry in remaining rows, switch to second row, use as a pivot to clear out the entries in the column below it. Continue (forward solving). When done, use pivots to clear out entries in column above the pivots (back-solving).

Variable in linear system corresponding to a pivot = **bound** variable; other variables = **free** variables

§4: Gaussian elimination: general procedure

The big fact: After elimination, the new system of linear equations have the exact **same solutions** as the old system. Because: row operations are reversible!
Reverse of E_{ij} is E_{ij}; reverse of $E_{ij}(m)$ is $E_{ij}(-m)$; reverse of $E_i(m)$ is $E_i(1/m)$
So: you can get old equations from new ones; so solution to new equations **must** solve old equations **as well**.

Reduced row form: apply elementary row operations so turn matrix A into one so that
(a) each row looks like $(000\cdots 0 * * \cdots *)$; firsdt $*$ = leading entry
(b) leading entry for row below is further to the right
Reduced row **echelon** form: in addition, have
(c) each leading entry is $= 1$
(d) each leading entry is the only non-zero number in its column.

RRF can be achieved by forward solving; RREF by back-solving and $E_i(m)$'s
Elimination: every matrix can be put into RREF by elementary row operations.
Big Fact: If a matrix A is put into RREF by two different sets of row operations, you get the **same matrix**.
RREF of an augmented matrix: can read off solutions to linear system.

$$\left(\begin{array}{cccc|c} 1 & 0 & 2 & 0 & 2 \\ 0 & 1 & 1 & 0 & 1 \\ 0 & 0 & 0 & 1 & 3 \end{array}\right)$$

means $x_4=3$, $x_2=1-x_3$
$x_1=2-2x_3$; x_3 is free

Inconsistent systems: row of zeros in coefficient matrix, followed by a non-zero number (e.g., 2). Translates as 0=2 ! System has no solutions.

Rank of a matrix = r(A) = number of non-zero rows in RREF = number of pivots in RREF.
Nullity of a matrix = n(A) = number of columns without a pivot = # columns − # pivots
rank = number of bound variables, nullity = number of free variables
rank ≤ number of rows, number of columns (at most one pivot per row/column!)
rank + nullity = number of columns = number of variables

A = coefficient matrix, $\tilde{A}$ = augmented matrix ($A = m \times n$ matrix)
system is consistent if and only if r(A) = r($\tilde{A}$)
r(A)=n : unique solution ; r(A)< n : infinitely many solutions

Chapter 2: Matrix algebra

§1: Matrix addition and scalar multiplication

Idea: take our ideas from vectors. Add entry by entry. Constant multiple of matrix: multiply entry by entry.
0 = matrix all of whose entries are 0
Basic facts:

A+B makes sense only if A and B are the same size (m×n) matrix
A+B = B+A
(A+B)+C = A+(B+C)
A+**0** = A
A+(-1)A = **0**
cA has the same size as A
c(dA) = (cd)A
(c+d)A = cA + dA
c(A+B) = cA + cB
1A = A

§2: Matrix multiplication

Idea: **don't** multiply entry by entry! We want matrix multplication to allow us to write a system of linear equations as Ax=b ....
Basic step: a row of A, times x, equals an **entry** of Ax. (row vector $(a_1, \ldots, a_n)$ times column vector $(x_1, \ldots, x_n)$ is $a_1x_1 + \cdots + a_nx_n$ ....) Thisa leads to:
In AB, each row of A is 'multiplied' by each column of B to obtain an entry of AB. Need: the length of the rows of A (= number of columns of A) = length of columns of B (= number of rows of B). I.e, in order to multiply, A must be m×n, and B must be n×k; AB is then m×k.

Formula: (i,j)th entry of AB is $\Sigma_{k=1}^{n} a_{ik}b_{kj}$

I = identity matrix; square matrix (n×n) with 1's on diagonal, 0's off diagonal
Basic facts:

AI = A = IA
(AB)C = A(BC)
c(AB) = (cA)B = A(cB)
(A+B)C = AC + BC
A(B+C) = AB + AC

In general, however it is **not** **not** true that AB and BA are the same; they are almost always different! ****

§3: **Applications of matrix arithmetic**

Ax=b ; A m-byn matrix. Think: x=vector=variable (size n) , Ax = vector = image of x (size m)

i.e., A takes vectors in R^n and spits out vectors in R^m; it's a function (which we call T_A) from R^n to R^m. More than that, it's a **linear** function:

$T_A(a\mathbf{x}+b\mathbf{y} = aT_A(\mathbf{x})+bT_A(\mathbf{y})$

With this new notation, matrix multiplication **becomes** composition of functions.

What do we **do** with matrix multiplication? Solve equations!

Ax=b ; basic idea, try to find a matrix B with BA=I, so then x = Ix = (BA)x = B(Ax) = Bb **solves** the equation. (How to **find** B? Wait.....)

Another application: Markov chains

Idea: in any give month, a fixed percentage people using one product switch to another.

$$a_1 = .3a_0 + .4b_0 + .2c_0 \ , \ b_1 = .4a_0 + .5b_0 + .6c_0 \ , \ c_1 = .3a_0 + .1b_0 + .2c_0$$

New distribution, given initial distribution x, is Ax, where

$$A = \begin{pmatrix} .3 & .4 & .2 \\ .4 & .5 & .6 \\ .3 & .1 & .2 \end{pmatrix}$$

More generally, a Markov chain consists of an (initial) probability distribution vector (entries are ≥ 0 and add up to 1) and a transition matrix A (entries are ≥ 0 and each column adds up to 1). The distribution evolves by multiplication by A. E.g, after 20 iterations, initial vector x evolves into $A^{20}x$.

§4: **Special matrices and transposes**

Elementary matrices:

A row operation (E_{ij} , $E_{ij}(m)$, $E_i(m)$) applied to a matrix A corresponds to multiplication (on the left) by a matrix (also denoted E_{ij} , $E_{ij}(m)$, $E_i(m)$) The matrices are obtained by applying the row operation to the identity matrix I_n. E.g., the 4×4 matrix $E_{13}(-2)$ looks like I, except it has a -2 in the (1,3)th entry.

The idea: if A $\rightarrow$ B by the elementary row operation E, then B = EA.

So if A $\rightarrow$ B $\rightarrow$ C by elementary row operations, then C = E_2E_1A ....

Row reduction **is** matrix multiplication!

A scalar matrix A has the same number c in the diagonal entries, and 0's everywhere else (the idea: AB = cB)

A diagonal matrix has all entries zero off of the (main) diangonal

A upper triangular matrix has entries =0 below the diagonal, a lower triangular matrix is 0 above the diagonal. A triangular matrix is either upper or lower triangular.

A strictly triangular matrix is triangular, and has zeros **on** the diagonal, as well. They come in upper and lower flavors.

The **transpose** of a matrix A is the matrix A^T whose columns are the rows of A (and vice versa). A^T is A reflected across the main diagonal. $(aij)^T = (aji)$; $(\text{m}\times\text{n})^T = (\text{n}\times\text{m})$

Basic facts:

$(A+B)^T = A^T + B^T$
$(AB)^T = B^T A^T$
$(cA)^T = \mathrm{c}A^T$
$(A^T)^T = \mathrm{A}$

Transpose of an elementary matrix is elementary:
$E_{ij}^T = E_{ij}$, $E_{ij}(m)^T = E_{ji}(m)$, $E_i(m)^T = E_i(m)$
A matrix A is **symmetric** if $A^T = A$
An occasionally useful fact: AE, where E is an elementary matrix, is the result of an elementary **column** operation on A .

The transpose and rank:
For any pair of compatible matrices, r(AB) $\leq$ r(A)
Consequences: r(A^T) = r(A) for any matrix A; r(AB) $\leq$ r(B), as well.

§5: **Matrix inverses**

One way to solve Ax=b : find a matrix B with BA=I . When is there such a matrix? (Think about square matrices...) A an n-by-n matrix ; n=r(I)=r(BA)$\leq$r(A)$\leq$n implies that r(A)=n . This is necessary, and it is also sufficient!
r(A)=n, then the RREF of A has n pivots in n rows and columns, so has a pivot in every row, so the RREF of A is I. But! this means we can get to I from A by row operations, which correspond to multiplication by elementary matrices. *So* multiply A (on the left) by the right elementary matrices and you get I; call the product of those matrices B and you get BA=I !
It turns out (by using the transpose) that AB=I as well!
A matrix B is an **inverse** of A if AB=I and BA=I; it turns out, the inverse of a matrix is always unique. We call it A^{-1} (and call A invertible).
Finding A^{-1} : row reduction! (of course...)
Build the "super-augmented" matrix (A|I) (the matrix A with the identity matrix next to it). Row reduce A, and carry out the operations on the entire row of the S-A matrix (i.e., carry out the identical row operations on I). Wnem done, if invertible+ the left-hand side of the S-A matrix will be I; the right-hand side will be A^{-1} !
I.e., if (A|I) $\rightarrow$ (I|B) by row operations, then I=BA .

Basic facts:

$(A^{-1})^{-1} = A$
if A and B are invertible, then so is AB, and $(AB)^{-1}$ - $B^{-1}A^{-1}$
$(cA)^{-1} = (1/c)A^{-1}$
$(A^T)^{-1} = (A^{-1})^T$
If A is invertible, and $AB = AC$, then $B = C$; if $BA = CA$, then $B = C$.

Inverses of elementary matrices:
$E_{ij}^{-1} = E_{ij}$, $E_{ij}(m)^{-1} = E_{ij}(-m)$, $E_i(m)^{-1} = E_i(1/m)$
Highly useful formula: for a 2-by-2 matrix,

$$A=\begin{pmatrix} a & b \\ c & d \end{pmatrix} \text{ and } D=ad-bc \text{ , } A^{-1}=\frac{1}{D}\begin{pmatrix} d & -b \\ -c & a \end{pmatrix}$$

(Note: need D=ad-bc $\neq 0$ for this to work....)
Some conditions for/consequences of invertibility: the following are all equivalent (A = n-by-n matrix).

1. A is invertible,
2. r(A) = n.
3. The RREF of A is I_n.
4. Every linear system Ax=b has a unique solution.
5. For one choice of b, Ax=b has a unique solution (i.e., if one does, they all do...).
6. The equation Ax=0 has only the solution x=0.
7. There is a matrix B with BA=I.

The euivalence of 4. and 6. is sometimes stated as **Fredholm's alternative**: Either every equation Ax=b has a unique solution, or the equation Ax=0 has a **non-trivial** solution (and only one of tyhe alternatives can occur).