Math 325 Topics Sheet for Exam 1

Throughout, we rely on set notation to make our ideas precise. Sets are typically described as the collection of all objects from a specific 'universe' that meet certain specific conditions:

E.g., $S = \{x \in \mathbb{Q} \mid x^2 < 2\}$ = the rational numbers (universe!) whose square is less than 2 (condition!).

$A \subseteq B$ means that every element of A is also an element of B; $A = B$ is typically established by showing that $A \subseteq B$ and $B \subseteq A$.

$A \cup B = \{x \mid x \in A \text{ or } x \in B\}$; $A \cap B = \{x \mid x \in A \text{ and } x \in B\}$; $A^c = \{x \mid x \notin A\}$

Functions: a function $f : A \to B$ is a rule which assigns to each $x \in A$ (the *domain*) exactly one element $f(x) \in B$ (the *codomain*).

We will mostly focus on functions with $A \subseteq \mathbb{R}$ and $B \subseteq \mathbb{R}$. Functions come in different flavors:
increasing: if $x > y$ then $f(x) > f(y)$. non-decreasing: $x \geq y$ implies $f(x) \geq f(y)$.
decreasing: if $x > y$ then $f(x) < f(y)$. non-increasing: $x \geq y$ implies $f(x) \leq f(y)$.
One-to-one: if $f(x) = f(y)$, then $x = y$. Alternate form: if $x \neq y$ then $f(x) \neq f(y)$. Third form!: for any $y \in B$, there is at most one $x \in A$ with $f(x) = y$.
Onto: for every $y \in B$, there is at least one $x \in A$ with $f(x) = y$.
One-to-one and onto = a one-to-one correspondence (or *bijection*).

Note: one-to-one and onto have a lot to do with what the domain and codomain of the funciton f are!

Composition: $f : A \to B$, $g : B \to C$, then $g \circ f : A \to C$ is $(g \circ f)(x) = g(f(x))$

If $g \circ f$ is onto, then g is onto! If $g \circ f$ is one-to-one, then f is one-to-one!

The Real Line: Nearly everything we will do comes down to understanding the properties of the real line $\mathbb{R}$.

$\mathbb{R}$ = a *complete ordered field*

Field: we have addition + and multiplication $\cdot$, so that
(A1) addition exists: if $x, y \in \mathbb{R}$ then $x + y \in \mathbb{R}$
(A2) commutativity: $x + y = y + x$ for every $x, y \in \mathbb{R}$
(A3) associativity: $x + (y + z) = (x + y) + z$ for every $x, y, z \in \mathbb{R}$
(A4) zero exists: there is $0 \in \mathbb{R}$ so that $x + 0 = x$ for every $x \in \mathbb{R}$
(A5) additive inverses exist: for every $x \in \mathbb{R}$ there is a $(-x) \in \mathbb{R}$ with $x + (-x) = 0$
(M1) multiplication exists: if $x, y \in \mathbb{R}$ then $x \cdot y \in \mathbb{R}$
(M2) commutativity: $x \cdot y = y \cdot x$ for every $x, y \in \mathbb{R}$
(M3) associativity: $x \cdot (y \cdot z) = (x \cdot y) \cdot z$ for every $x, y, z \in \mathbb{R}$
(M4) one exists: there is $1 \in \mathbb{R}$ so that $x \cdot 1 = x$ for every $x \in \mathbb{R}$
(M5) multiplicative inverses exist:
for every $x \in \mathbb{R}$ with $x \neq 0$ there is a $x^{-1} \in \mathbb{R}$ with $x \cdot x^{-1} = 1$
(D) distributivity: for every $x, y, z \in \mathbb{R}$ we have $x(y + z) = xy + xz$

Ordered: there is a collection $\mathcal{P}$ = the *positive numbers* so that
(P1) closed under addition: if $x, y \in \mathcal{P}$ then $x + y \in \mathcal{P}$
(P2) closed under multiplication: if $x, y \in \mathcal{P}$ then $x \cdot y \in \mathcal{P}$
(P3) trichotomy:
for every $x \in \mathbb{R}$ *exactly one* of the following is true: $x \in \mathcal{P}$, or $-x \in \mathcal{P}$, or $x = 0$.

We then defined the ordering $x < y$ to mean that $y - x \in \mathcal{P}$. (So $x > 0$ means $x \in \mathcal{P}$.)
Basic properties:
trichotomy! for any $x, y \in \mathbb{R}$ exactly one of $x < y$, $x = y$, or $x > y$ is true.
transitivity: $x < y$ and $y < z$ implies that $x < z$
$x < y$ implies that $x + z < y + z$ for any $z \in \mathbb{R}$
$x < y$ and $z > 0$ implies that $xz < yz$ (because $yz - xz = (y - x)z$ with $y - x, z \in \mathcal{P}$)
weak inequality: $a \leq b$ means $a < b$ or $a = b$; similar properties hold!

From these basic properties we can recover many familiar properties we have seen before; for example
$(-x)y = -(xy)$ $\quad$ $(-x)(-y) = xy$ $\quad$ $-(-x) = x$ $\quad$ $(x^{-1})^{-1} = x$
$x < 0$ and $y > 0$ implies $xy < 0$ $\quad$, $\quad$ $z < 0$ and $w < 0$ implies $zw > 0$
the additive inverse of a number is unique (i.e., if $x + y = 0 = x + z$ then $y = z$)

Absolute value: $|x| = x$ if $x \geq 0$, otherwise it is $-x$. So $|x| \geq 0$ for all x.
"Triangle inequality": $|x + y| \leq |x| + |y|$ for every $x, y \in \mathbb{R}$.
[Useful 'opposite' consquence: $|x - y| \geq |x| - |y|$
(useful for showing that $|x - y|$ is not small!]
$|x - y|$ = the distance between x and y. Triangle inequality: $|x - z| \leq |x - y| + |y - z|$

Proving things: Our ultimate goal is to provide proofs of some of the important results from calculus. This means that we need to justify the assertions we make, showing how a hypothesis forces our colnclusions to be true. Two often-used approaches:

Case analysis: Starting from a hypothesis (e.g., $x \neq 0$), one of several possibilities (cases) must be true (e.g., $x > 0$ or $x < 0$). If we show that in each case our hoped-for conclusion is true (e.g., $x^2 > 0$), then the hypothesis implies the conclusion ($x \neq 0$ implies $x^2 > 0$).

Proof by contradiction: "A implies B" is the same as "it is not possible for A to be true and also that B is false". Proof by contradiction consists of starting from 'A is true and B is false' and showing that we must inevitably show that something we know is false is true. This means that we cannot have A true and B false; so A implies B !

Example: using the Rational Roots Theorem (see below) we can show that it is not possible to have $x^3 = 5$ and $x \in \mathbb{Q}$. So $x^3 = 5$ implies $x \notin \mathbb{Q}$.

Another approach we will often use: induction! (see below)

Completeness: From the natural (= counting) numbers $\mathbb{N}$ we get the integers $\mathbb{Z}$ (by taking additive inverses) and then the rationals $\mathbb{Q}$ (by taking multiplicative inverses). But to get the reals $\mathbb{R}$ we need to step beyond the properties above.

A set $A \subseteq \mathbb{R}$ is *bounded (bdd) from above* if there is a $M \in \mathbb{R}$ so that $x \leq M$ for every $x \in A$.
A *least upper bound* λ is an upper bound for A so that no *smaller* number is an upper bound. In symbols: $x \leq \lambda$ for every $x \in A$ and if $\mu < \lambda$ then there is an $x \in A$ with $\mu < x$
[Equivalently: λ is an upper bd for A and if ν is *also* an upper bound for A then $\lambda \leq \nu$.]

Completeness Axiom: Every *non-empty* set $A \subseteq \mathbb{R}$ that is bdd from above *has* a least upper bound.

Least upper bound of A is unique! $\lambda = \text{lub}(A)$

Application: If $x, y \in \mathbb{R}$ and $y - x > 1$, then there is an $n \in \mathbb{Z}$ with $x \leq n < y$.

Application: $A = \{x \in \mathbb{R} \ : \ x^2 < 2\}$ is non-empty and bdd above: $\lambda = \text{lub}(A)$. Then we showed: $\lambda^2 = 2$ (!) So λ = what we would call $\sqrt{2}$

Rational Roots Theorem: If $p(x) = a_0x^n + \cdots a_{n-1}x + a_n$ is a polynomial with integer coefficients $a_i \in \mathbb{Z}$ for all i), and if $r = \alpha/\beta$ is a rational root of p ($p(\alpha/\beta) = 0$ where α and β have no factors in common, *then* α evenly divides a_n and β evenly divides a_0.

Since $\sqrt{2}$ is a root of $p(x) = x^2 - 2$, which by the rational roots theorem has no rational roots, $\sqrt{2} \notin \mathbb{Q}$. (!) By the same reasoning, if $n \in \mathbb{N}$ and $\sqrt{n} \in \mathbb{Q}$ then $\sqrt{n} \in \mathbb{N}$.

Well-ordering Property: If $A \subseteq \mathbb{N}$ is non-empty, then it has a *smallest element*. That is, there is an $\lambda \in A$ so that $lambda \leq z$ for every $z \in A$. (In general for a set A whose greatest lower bound λ lies in A, we call it the *mminimum* of A. If the least upper bound lies in A, we call it the *maximum*.)

Application: the *Principle of Mathematical Induction:* If $A \subseteq \mathbb{N}$ is a set satisfying (1) $n_0 \in A$, and (2) if $n \geq n_0$ and $n \in A$ then $n+1 \in A$, *then* $\{n \in \mathbb{N} \ : \ n \geq n_0\} \subseteq A$.

[Why? If not all $n \geq n_0$ are in A, then there is a smallest n which is not in A. But then $n-1 \in A$ (or $n = n_0$) both of which contradict our hypotheses!]

PMI as it is usually stated: If $P(n)$ is a statement about the integer n so that

(1) $P(n_0)$ is true, and

(2) if $n \geq n_0$ and $P(n)$ is true then $P(n+1)$ must also be true,

then $P(n)$ is true for every integer $n \geq n_0$.

Sample applications:

For every integer $n \geq 1$ we have $\displaystyle\sum_{k=1}^{n} k \ = \ \frac{n(n+1)}{2}$

For every integer $n \geq 1$ we have $\displaystyle\sum_{k=1}^{n} k^2 \ = \ \frac{n(n+1)(2n+1)}{6}$

For every $n \geq 5$ we have $2^n > n^2$

Fibonacci sequence: $a_0 = a_1 = 1$ and for $n \geq 1$ we have $a_{n+1} = a_n + a_{n-1}$. Then for every $n \geq$ (something) we have $a_n \geq \left(\frac{3}{2}\right)^n$, and for every $n \geq 12$ we have $a_n \geq n^2$.

[For all of these the method by which we established the inductive step is probably more important as we go forward than the actual result!]

More Completeness property consequences:

If $\epsilon > 0$ then there is an $n \in \mathbb{N}$ so that $n\epsilon > 0$ (the Archimedean Principle).

[Add a positive number to itself enough times and you will get a big number!]

Rationals are everywhere: If $x, y \in \mathbb{R}$ with $x < y$ then there is an $r \in \mathbb{Q}$ so that $x < r < y$.

Limits of functions: Calculus is all about studying functions $f : \mathbb{R} \to \mathbb{R}$ or more generally $f : D \to \mathbb{R}$ for some domain in $\mathbb{R}$. And the fundamanetal concept the distinguishes calculus from what comes before it is the idea of a limit of a function.

$\lim_{x \to a} f(x) = L$ means that $|f(x) - L|$ is small, so long as $|x - a|$ is small enough. One feature: a need not be in the domain of f. If fact, even if is is, we do not care what value f takes there; our formal definition of the limit is

For every $\epsilon > 0$, there is a $\delta > 0$ so that $x \in D$ and $0 < |x - a| < \delta$ implies that $|f(x) - L| < \epsilon$.

One sticky point: such limits need not be unique! [If $x \in D$ and $0 < |x - a| < \delta$ is satified by no number, then L could be anything we want!] For that matter, the limit need not exist! However, we showed that if f is defined on all of $\mathbb{R}$ (or everywhere but at a) and the limit as we approach a exists, then that limit is unique. If it does, we say that f converges at a. A useful shorthand: $f(x) \to L$ as $x \to a$.

Proving that $f(x) \to L$ as $x \to a$ is a kind of two-person game: no matter what $\epsilon > 0$ someone gives us, we need to demonstrate that there is a $\delta > 0$ so that $0 < |x - a| < \delta$ will guarantee that $|f(x) - L| < \epsilon$. Usually we accomplish this by 'finding' $|x - a|$ in the expression $|f(x) - L|$ and arguing that by building a δ out of ϵ, L, a, and f we can guarantee that $|x-a| < \delta$ forces $|f(x)-L|$ to remain small. As an example, $4x-1 \to 7$ as $x \to 2$ because $|(4x-1)-7| = 4|x-2|$, so $|x-2| < \epsilon/4 = \delta$ implies that $|(4x-1)-7| < \epsilon$, so we choose $\delta < \epsilon/4$ when challenged with $\epsilon > 0$.

A common technique that we use in these sorts of arguments is that if $L \neq M$ then $|x-M| = |(x-L)+(L-M)|$ and so $|x-M| \leq |x-L|+|L-M| < \delta+|L-M|$ cannot get too large, while $|M-L| \leq |M-x|+|x-L|$ so $0 < |M-L|-\delta < |M-L|-|x-L| \leq |x-M|$ (when δ is small enough!), so $|x - M|$ also cannot get too small! [This is useful when a term like $|x - M|$ appears in the denomenator of an $|f(x) - L|$ calculuation...]

Most of our familiar results about limits of functions from calculus can be verified using this more rigorous framework. [This was, after all, why we were building a rigorous framework!] For example, if $f(x) \to L$ and $g(x) \to M$ as $x \to a$, and k is a konstant, then $(f+g)(x) \to L + M$, $f(x) - g(x) \to L - M$, $kf(x) \to kL$, $(f \cdot g)(x) \to LM$, and (so long as $M \neq 0$ $f(x)/g(x) \to L/M$. Each of these required us to manufacture a $\delta > 0$ to control, for example, $|f(x)g(x) - LM| = |(f(x) - L)g(x) + (g(x) - M)L| \leq |f(x) - L| \cdot |g(x)| + |g(x) - M| \cdot |L|$, which we can do by making sure that $|g(x)|$ does not get too large!

We can quickly show that for $f(x) = k$=konstant and $f(x) = x$, we have $k \to k$ and $x \to a$ as $x \to a$. These, together with the sum and product results above, and induction, lead us to the result that for a polynomial $p(x)$ with coeffficients in $\mathbb{R}$, we have $p(x) \to p(a)$ as $x \to a$.

Continuity: From calculus you are used to the idea that for many functions to compute its limit we "plug in". That is, $\lim_{x\to c} f(x) = f(c)$. We call such a function *continuous at c*. If it is not continuous at c, we say it is *discontinuous* at c. If $f : D \to \mathbb{R}$ is continuous at c for every $c \in D$, we say that it is continuous on D. Continuity can be described using ϵ's and δ's: f is continuous at c provided that for every $\epsilon > 0$ there is a $\delta > 0$ so that $|x - c| < \delta$ implies that $|f(x) - f(c)| < \epsilon$.

Because continuity has to do with limits, and we understand how limits behave when we combine functions, we can deduce some general rules about how continuity behanves when we combine functions: the sum, difference, constant multiple, product and (when we avoid where the denomenator is zero) quotient of continuous functions is itself continuous.