A selection of Exam 3 practice problem solutions.

1. Define the sequence a_n inductively, by $a_1 = 1$ and, for $n \geq 2$, $a_n = n + a_{n-1}$. Show that the statement

$$P(n) : \text{“ } a_n = \frac{(n+3)(n-2)}{2} \text{ ”}$$

satisfies $P(n)$ is true $\Rightarrow$ $P(n+1)$ is true; show, however, that $P(n)$ is not always true! Why does this not violate the Principle of Mathematical Induction?

If $a_n = \frac{(n+3)(n-2)}{2}$ then

$$\begin{aligned} a_{n+1} &= (n+1) + a_n \\ &= (n+1) + \frac{(n+3)(n-2)}{2} \\ &= \frac{(n+3)(n-2) + 2(n+1)}{2} \\ &= \frac{n^2+3n-2n-6+2n+2}{2} \\ &= \frac{n^2+3n-4}{2} = \frac{(n+4)(n-1)}{2} \\ &= \frac{((n+1)+3)((n+1)-2)}{2} \end{aligned}$$

So $P(n)$ true implies $P(n+1)$ true.

But! $a_1 = 1$ and

$$\frac{(1+3)(1-2)}{2} = \frac{4(-1)}{2} = -2$$

So $P(1)$ is false!

This is not a problem, since PMI requires $P(n) \Longrightarrow P(n+1)$ and $P(n_0)$ true for some $n_0 \in \mathbb{N}$ (e.g. $P(1)$, $n_0 = 1$). So the hypotheses of PMI is not satisfied.

4. Show using the "epsilon-delta" formulation of the limit, that $\lim_{x\to 2} x^3 - x^2 + 2x + 1 = 9$.

We want: Given $\varepsilon > 0$, to find a $\delta > 0$ so that $0 < |x-2| < \delta$ implies $|(x^3-x^2+2x+1)-9| < \varepsilon$

But: $|(x^3-x^2+2x+1)-9| = |x^3-x^2+2x-8|$

$= |(x-2)(x^2+x+4)| = |x-2||x^2+x+4|$

To make this small (since $|x-2|$ can be assumed small) we need $|x^2+x+4|$ is ~~not~~ not big. But if $|x-2|<1$, say, then $-1<x-2<1$ so $1<x<3$, so

$1 = 1\cdot 1 < 1\cdot x < x^2 < 3\cdot x < 3\cdot 3 = 9$, so

$2 = 1+1 < x^2+1 < x^2+x < x^2+3 < 9+3 = 12$, so

$6 = 2+4 < x^2+x+4 < 12+4 = 16$, so $6 < x^2+x+4 < 16$

so $-16 < x^2+x+4 < 16$, so $|x^2+x+4| < 16$.

So set $\delta = \min\{1, \frac{\varepsilon}{16}\}$, then $0<|x-2|<\delta$ implies $|x-2|<1$ so $|x^2+x+4|<16$, so

$|(x^3-x^2+2x+1)-9| = |x-2||x^2+x+4| < |x-2|\cdot 16$

$< \frac{\varepsilon}{16}\cdot 16 = \varepsilon$, as desired. ///

6. Show that if $f : \mathbb{R} \to \mathbb{R}$ is **uniformly continuous**, then the sequence of functions $f_n : \mathbb{R} \to \mathbb{R}$ defined by $f_n(x) = f(x + \frac{1}{n})$ converges **uniformly** to f.

f unif cts, so $\forall \varepsilon > 0\ \exists \delta > 0$ s.t.
$|x-y| < \delta$ implies $|f(x) - f(y)| < \varepsilon$.

we want, for $\varepsilon > 0$, an $N \in \mathbb{N}$ so that $n \geq N \Rightarrow$
$|f_n(x) - f(x)| < \varepsilon$ for all $x \in \mathbb{R}$. But!
$|f_n(x) - f(x)| = |f(x + \frac{1}{n}) - f(x)|$, so to ensure this
is $< \varepsilon$ we ~~ne~~ only need to have $|(x + \frac{1}{n}) - x| = |\frac{1}{n}| < \delta$
for all $x \in \mathbb{R}$. But if we pick $N \in \mathbb{N}$ so that
$\frac{1}{N} < \delta$ (i.e. $N > \frac{1}{\delta}$), then $n \geq N$ implies that
$\frac{1}{n} \leq \frac{1}{N} < \delta$, so $|f_n(x) - f(x)| = |f(x + \frac{1}{n}) - f(x)| < \varepsilon$,
as desired.

so $f_n \to f$ uniformly. ///

B2: $\alpha^2 = \alpha + \sqrt{3} \Rightarrow \alpha^2 - \alpha = \sqrt{3} \Rightarrow$

$(\alpha^2-\alpha)^2 = 3 = \alpha^4 - 2\alpha^3 + \alpha^2 \Rightarrow \underline{\alpha^4 - 2\alpha^3 + \alpha^2 - 3 = 0}$

$f(x) = x^4 - 2x^3 + x^2 - 3$ is cts on $[1,2]$ and

$f(1) = 1 - 2 + 1 - 3 = -3 < 0$

$f(2) = 16 - 16 + 4 - 3 = 1 > 0$

so by IVT there is an $\alpha \in [1,2]$ so that $f(\alpha) = 3$

so $(\alpha^2-\alpha)^2 = 3$. But note that since $\alpha \in [1,2]$,

$\alpha^2 - \alpha = \alpha(\alpha - 1) \geq 1 \cdot 0$ (since $\alpha - 1 \geq 0$). So $\alpha^2 - \alpha$ is a

positive number with $(\alpha^2-\alpha)^2 = 3$, so $\alpha^2 - \alpha = \sqrt{3}$.

So $\alpha^2 = \alpha + \sqrt{3}$.

There is no rational number with $\alpha^2 = \alpha + \sqrt{3}$, since if there were then $\alpha \in \mathbb{Q}$ and $f(\alpha) = 0$. But the rational roots thm says that then $\alpha = p/q$ with p dividing -3 and q dividing 1, so $\alpha = 1, -1, 3,$ or -3. But $f(1) = f(-1) = -3 \neq 0$ and $f(3) = f(-3) = \cancel{81} \; 81 - 54 + 9 - 3 = 33 \neq 0$. So no rational number has $\alpha^2 = \alpha + \sqrt{3}$. ∎

B3. To make f cts at $x=0$ we need $\lim_{x\to 0} f(x) = a$.

But by L'Hôpital,

$$\lim_{x\to 0} f(x) = \lim_{x\to 0} \frac{\sin x - \tan x}{x^2} = \lim_{x\to 0} \frac{\cos x - \sec^2 x}{2x} = \lim \frac{\sin x - 2\sec x \sec x \tan x}{2}$$

$$= \lim_{x\to 0} \tfrac{1}{2}\left(\sin x - \frac{2\sin x}{\cos^3 x}\right) = \tfrac{1}{2}\left((0) - \frac{2(0)}{1^3}\right) = 0 \;(!)$$

So $f(0) = 0$ makes f cts. Then

$$f'(0) = \lim_{x\to 0} \frac{f(x) - f(0)}{x - 0} = \lim_{x\to 0} \frac{\sin x - \tan x}{x^3} = \lim \frac{\cos x - \sec^2 x}{3x^2}$$

$$= \lim_{x\to 0} \frac{\sin x - \frac{2\sin x}{\cos^3 x}}{6x} = \lim_{x\to 0} \tfrac{1}{6}\left(1 - \frac{2}{\cos^3 x}\right)\left(\frac{\sin x}{x}\right)$$

$$= \lim_{x\to 0} \tfrac{1}{6}\left(1 - \frac{2}{\cos^3 x}\right) \lim_{x\to 0} \frac{\sin x}{x} = \tfrac{1}{6}\left(1 - \frac{2}{1^3}\right) \lim_{x\to 0} \frac{\cos x}{1}$$

$$= \tfrac{1}{6}\left(1 - \frac{2}{1^3}\right)\left(\frac{1}{1}\right) = \tfrac{1}{6}(-1) = -1/6 .$$

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B4. Since $x_{k-1} \le c_k, d_k \le x_k$ and $R(f, P, \{c_k\}) \le R(g, P, \{d_k\})$, then if $L = \int_a^b f(x)\,dx > \int_a^b g(x)\,dx = M$, setting $\varepsilon = L - M$ we can find a P so that $|R(f, P, \{c_k\}) - L| < \varepsilon/2$ and $|R(f, P, \{d_k\}) - M| < \varepsilon/2$ (by finding ~~partitions~~ a $\delta > 0$ so that $\|P\| < \delta$ always works)

But then $L - \varepsilon/2 < R(f, P, \{c_k\}) \le R(g, P, \{d_k\}) < M + \varepsilon/2$, and

So $L - M < \varepsilon/2 + \varepsilon/2 = \varepsilon$, so $\varepsilon < \varepsilon$ a contradiction.

So $L > M$ must be false, and $\int_a^b f(x)\,dx \le \int_a^b g(x)\,dx$.

5. Show that if $f, g : [a,b] \to \mathbb{R}$ are a pair of bounded functions, and $U(h)$ denotes the upper Riemann integral of h over the interval $[a,b]$, then

$$U(f+g) \leq U(f) + U(g) .$$

We wish to show that the sup of $U(f+g,P)$, taken over all partitions of $[a,b]$, is $\leq U(f)+U(g)$. We can establish this by showing that $U(f)+U(g)$ is an <u>upper</u> <u>bound</u> for the set $\{U(f+g,P) \ : \ P$ a partition of $[a,b]\}$, that is, $U(f+g,P) \leq U(f)+U(g)$ for every partition P.

But we know that $U(f,Q) \leq U(f)$ and $U(g,R) \leq U(g)$ for any partitions Q, R of $[a,b]$, since $U(f)$ and $U(g)$ are the suprema of such upper Riemann sums. So in particular we have $U(f,P) \leq U(f)$ and $U(g,P) \leq U(g)$.

Our intended result then follows if we show that $U(f+g,P) \leq U(f,P)+U(g,P)$, since then $U(f+g,P) \leq U(f,P)+U(g,P) \leq U(f)+U(g)$, establishing that $U(f)+U(g)$ is an upper bound for the $U(f+g,P)$, as desired. But $U(f+g,P) \leq U(f,P)+U(g,P)$ follows from the fact that

$U(f+g,P) = \sum_i \sup\{(f+g)(x) \ : \ x \in [x_i,x_{i+1}]\}(x_{i+1}-x_i)$

and, for each i, we have

$$\sup\{(f+g)(x) \ : \ x \in [x_i,x_{i+1}]\} \\ \leq \sup\{f(x) \ : \ x \in [x_i,x_{i+1}]\} + \sup\{g(x) \ : \ x \in [x_i,x_{i+1}]\} = F+G \ ;$$

this is, effectively, from an old problem set, although we can see this directly, since $f(x) \leq A$ and $g(x) \leq B$ for every $x \in [x_i,x_{i+1}]$ (since they are suprema), so $(f+g)(x) = f(x)+g(x) \leq A+B$ for every $x \in [x_i,x_{i+1}]$, making $A+B$ an upper bound, so it is $\geq$ the supremum.

Putting this all together, we find that since

$$\sup\{(f+g)(x) \ : \ x \in [x_i,x_{i+1}]\} \\ \leq \sup\{f(x) \ : \ x \in [x_i,x_{i+1}]\} + \sup\{g(x) \ : \ x \in [x_i,x_{i+1}]\},$$

for every i, we have

$$U(f+g,P) = \sum_i \sup\{(f+g)(x) \ : \ x \in [x_i,x_{i+1}]\}(x_{i+1}-x_i) \\ \leq \sum_i [\sup\{f(x) \ : \ x \in [x_i,x_{i+1}]\} + \sup\{g(x) \ : \ x \in [x_i,x_{i+1}]\}](x_{i+1}-x_i) \\ = \sum_i [\sup\{f(x) \ : \ x \in [x_i,x_{i+1}]\}](x_{i+1}-x_i) + \sum_i [\sup\{g(x) \ : \ x \in [x_i,x_{i+1}]\}](x_{i+1}-x_i) \\ = U(f,P)+U(g,P),$$

so $U(f+g,P) \leq U(f,P)+U(g,P)$ for every partition P, so $U(f+g,P) \leq U(f)+U(g)$ for every partition P, so $U(f+g) \leq U(f)+U(g)$.

C1. (wuth some names changed...) Show that if $a < b < c$ and if $f : [a,b] \to \mathbb{R}$ and $g : [b,c] \to \mathbb{R}$ are both continuous functions, and $f(b) = g(b)$, then the function $h : [a,c] \to \mathbb{R}$ defined by

$$h(x) = \begin{cases} f(x) & \text{if } x \leq b \\ g(x) & \text{if } x \geq b \end{cases}$$

is continuous at $x = b$. Why is it also continonuous at every <u>other</u> point in $[a,c]$?

To establish that h is continuous at $x = b$, we wish to show that for any $\epsilon > 0$ there is a $\delta > 0$ so that $|x-a| < \delta$ implies that $|h(x)-h(a)| < \epsilon$. But since f is countinuous at $x = b$ (which is the right endpoint of its interval of definition), we know that $\lim_{x \to b^-} f(x) = f(b)$, so for our $\epsilon > 0$ above, there is a $\delta_1 > 0$ so that $|x-b| < \delta+1$ <u>and</u> $x < b$ we have $|f(x)-f(b)| < \epsilon$. Also, since g is continuous at b (which is the left endpoint of its interval of definition), we know that $\lim_{x \to b^+} g(x) = g(b)$, so for our $\epsilon > 0$ above, there is a $\delta_2 > 0$ so that $|x - b| < \delta_2$ and $x > b$ we have $|g(x) - g(b)| < \epsilon$.

But since $f(b) = h(b) = g(b)$, and $h(x) = f(x)$ when $x < b$ and $h(x) = g(x)$ when $x > b$, we have actually established that if $|x - b| < \delta_1$ and $x < b$ then $|h(x) - h(b)| < \epsilon$, and if $|x - b| < \delta_2$ and $x > b$ then $|h(x) - h(b)| < \epsilon$. Note that if $x = b$, then $|h(x) - h(b)| = |h(b) - h(b)| = 0 < \epsilon$ automatically. So, if we set $\delta = \min\{\delta_1, \delta_2\} > 0$, then $|x - b| < \delta$ implies that $x = b$ or $|x - b| < \delta_1$ and $x < b$ or $|x - b| < \delta_2$ and $x > b$; in every case, we can conclude that $|h(x) - h(b)| < \epsilon$. SO we have found a $\delta > 0$ so that $|x - b| < \delta$ implies tht $|h(x) - h(b)| < \epsilon$. So h is continuous at $x = b$.

For every other point $d \in [a, c]$, either $d < b$ or $d > b$. If $d < b$, then $b - d = \delta_1 > 0$, and so $|x - d| < \delta_1$ implies that $x < b$, so $h(x) = f(x)$. So if we have an $\epsilon > 0$, then the continuity of f at $x = d$ implies that there is a $\delta_2 > 0$ so that $|x - d| < \delta_2$ and $x \in [a, b]$ implies that $|f(x) - f(d)| < \epsilon$. Then, setting $\delta = \min\{\delta_1, \delta_2\} > 0$, if $|x - d| < \delta$, then $|x - d| < \epsilon_1$ so $x < b$ and so $x \in [a, b]$, so $f(x) = h(x)$, and so since $|x - d| < \delta_2$, as have $|h(x) - h(d)| = |f(x) - f(d)| < \epsilon$. So h is continuous at $x = d$.

The case of $d > b$ is essentially identical. If $d > b$, then $d - b = \delta_1 > 0$, and so $|x - d| < \delta_1$ implies that $x > b$, so $h(x) = g(x)$. So if we have an $\epsilon > 0$, then the continuity of g at $x = d$ implies that there is a $\delta_2 > 0$ so that $|x - d| < \delta_2$ and $x \in [a, b]$ implies that $|g(x) - g(d)| < \epsilon$. Then, setting $\delta = \min\{\delta_1, \delta_2\} > 0$, if $|x - d| < \delta$, then $|x - d| < \epsilon_1$ so $x > b$ and so $x \in [b, c]$, so $g(x) = h(x)$, and so since $|x - d| < \delta_2$, as have $|h(x) - h(d)| = |g(x) - g(d)| < \epsilon$. So h is continuous at $x = d$.