Math 325 Problem Set 10

Starred (*) problems are due Friday, November 30.

(*) 59. Show that if f is integrable on $[a, b]$, and you can show (from the definition as $|\sum_{i=1}^{n} f(c_i)(x_i - x_{i-1}) - L| < \epsilon$) that $\int_a^b f(x)\, dx = L$ <u>and</u> $\int_a^b f(x)\, dx = M$, then $L = M$. [I.e., 'the value of an integral is unique'.]

[Suppose not! Show that there is a partition P that gets you into trouble...]

(*) 60. (Belding and Mitchell, p.129, #4b and #8)

(a): Show that if h is integrable on the interval $[a, b]$ and $h(x) \geq 0$ for every $x \in [a, b]$, then for every partition $P = \{a = x_0 < x_1 < \cdots < x_n = b\}$ of $[a, b]$ and set of 'samples' $S = \{c_1, \ldots, c_n\}$ with $c_i \in [x_{i-1}, x_i]$ for each i, we have the Riemann sum has $R(h, P, S) \geq 0$. Explain why we can then conclude that $\int_a^b h(x)\, dx \;\geq\; 0$.

(b): Use part (a) and the properties of integrals to show that if f and g are integrable on $[a, b]$ and $f(x) \geq g(x)$ for every $x \in [a, b]$, then $\int_a^b f(x)\, dx \;\geq \int_a^b g(x)\, dx$.

61. (Belding and Mitchell, p.129, #4(a)) Suppose that f is integrable on $[a, b]$, and $m \leq f(x) \leq M$ for every $x \in [a, b]$. Show that $m(b - a) \leq \int_a^b f(x)\, dx \leq M(b - a)$.

62. Suppose that f is differentiable on $[a, b]$ with $f'(x) < k$ for every $x \in [(a, b)$. Let P be any partition of $[a, b]$. Show that $U(f, P) - L(f, P) \leq k(b - a)^2$.

[The textbook provides some hints.]

63. Show: If f is integrable on $[a, b]$, then $g(x) = |f(x)|$ is integrable on $[a, b]$ and $|\int_a^b f(x)\, dx| \leq \int_a^b |f(x)|\, dx$.

[Hint: for integrable, use that $|f(y)| - |f(x)| \leq |f(y) - f(x)|$ for every x and y to show that $U(|f|, P) - L(|f|, P) \leq U(f, P) - L(f, P)$. For the inequality, look at earlier problems!]

(*) 64. (Belding and Mitchell, p.147, #1) Show, using the fundamental theorem of calculus, that for all $x \in \mathbb{R}$ we have $\int_0^x |t|\, dt \;=\; \frac{1}{2}x|x|$. [Consider the two cases $x > 0$ and $x < 0$ separately.]

65. (Belding and Mitchell, p148., #10) Prove the **generalized integral mean value theorem**: If f and g are continuous functions on $[a, b]$ and $g(x) > 0$ on $[a, b]$, then there is a $c \in [a, b]$ such that $\int_a^b f(x)g(x)\, dx = f(c) \int_a^b g(x)\, dx$.

[The textbook provides a suggestion of how to do this.]