Math 325 Problem Set 5

Starred (*) problems are due Friday, September 28.

24. Show that if $0 \le x < 1$ then for any $\epsilon > 0$ there is an $n \in \mathbb{N}$ so that $x^n < \epsilon$.

    [Hint: Suppose not! Then look at lower bounds for $A = \{x^n : n \in \mathbb{N}\}$.]

(*) 25. (Belding and Mitchell, p.63, #1) Using the ϵ-δ formulation of limits,

(*) (b) show that $\lim_{x \to -1} 1 - 2x = 3$.

(c) show that $\lim_{x \to 5} \sqrt{x+4} = 3$.

(*) (e) show that $\lim_{x \to -3} \dfrac{1}{8 - 4x} = \dfrac{-1}{4}$.

26. (Belding and Mitchell, p.63, #2) Suppose that $g : \mathbb{R} \to \mathbb{R}$ is a *bounded* function, i.e., there is an $M \in \mathbb{R}$ so that $|g(x)| \le M$ for all $x \in \mathbb{R}$. Show that $\lim_{x \to 0} xg(x) = 0$.

27. (Belding and Mitchell, p.64, #6) Show that if $\lim_{x \to a} f(x) = L$, then for any $\eta > 0$ there is a $\tau > 0$ so that $0 < |x_1 - a| < \tau$ and $0 < |x_2 - a| < \tau$ imply that $|f(x_1) - f(x_2)| < \eta$. [I.e., "if f has a limit as we approach a, then points close (enough) to a have values close to one another".]

    [N.B. This is a useful way to show that a function has *no* limit as x approaches a : show that this conclusion does not hold!]

(*) 28. (The 'Squeeze Play' Theorem) Suppose that $f, g, h : \mathbb{R} \to \mathbb{R}$, are functions with with $f(x) \le g(x) \le h(x)$ for all $x \in \mathbb{R}$ with $0 < |x - a| < M$ for some $a \in \mathbb{R}$ and $M > 0$. Suppose further that

    $\lim_{x \to a} f(x) = L = \lim_{x \to a} h(x)$. Show that $\lim_{x \to a} g(x) = L$.

    [See Belding and Mitchell, p.64, #8 for an outline that you might follow.]

(*) 29. (Belding and Mitchell, p.64, #9) If $f : \mathbb{R} \to \mathbb{R}$ is a function, $\lim_{x \to a} f(x) = L$, and for some $K, M \in \mathbb{R}$ with $M > 0$ we have $f(x) \le K$ for all x with $0 < |x - a| < M$, show that $L \le K$.

    [What's the alternative?]

30. (Belding and Mitchell, p.69, #1) Show, using our limit theorems and induction, that for every $n \in \mathbb{Z}$ we have $\lim_{x \to a} x^n = a^n$ (where we assume that $a \ne 0$ when $n < 0$).