Math 325 Problem Set 7

Starred (*) problems are due Friday, October 19.

(*) 38. (Belding and Mitchell, p.89, #10(a)) Show that if $a, b \in \mathbb{R}$, $a < b$, and $f : (a, b) \to \mathbb{R}$ is *uniformly* continuous, then f is bounded: there are $M, N \in \mathbb{R}$ so that $M \leq f(x) \leq N$ for every $x \in (a, b)$.

Hint: the argument that we gave in class sort of works, but we can't 'start' at a (it's not in the domain)!. 'Start' in the middle, instead!

39. (Belding and Mitchell, p.88, #5) Show that if $f, g : (a, b) \to \mathbb{R}$ are both uniformly continuous, then $f + g$ and $f - g$ (as functions from (a, b) to $\mathbb{R}$) are also uniformly continuous.

(*) 40. (Belding and Mitchell, p., #5, the other part) If $a, b \in \mathbb{R}$ and $f, g : (a, b) \to \mathbb{R}$ are uniformly continuous, show that $fg : (a, b) \to \mathbb{R}$ is uniformly continuous. Show, by example, on the other hand that $f, g : \mathbb{R} \to \mathbb{R}$ both uniformly continuous does <u>not</u> imply that $fg : \mathbb{R} \to \mathbb{R}$ is uniformly continuous.

[Hint: for the first, try to follow the 'usual' continuity argument for products; problem #38 will help! for the second, you don't need to get very creative...]

41. (Belding and Mitchell, p.89, #14) Let f and g be uniformly continuous on $\mathbb{R}$. Prove that their composition $f \circ g$ is also uniformly continuous on $\mathbb{R}$.

42. (Belding and Mitchell, p.89, #12) A function $f : D \to \mathbb{R}$ is called *Lipschitz* (with Lipschitz constant $M \in \mathbb{R}$ if for every $x, y \in D$ we have $|f(x) - f(y)| \leq M \cdot |x - y|$. Show that a Lipschitz function is uniformly continuous on D. Use this to show that the function $f : \mathbb{R} \to \mathbb{R}$ defined by

$$f(x) = \frac{1}{1 + |x|}$$

is uniformly continuous on $\mathbb{R}$.

43. Suppose that $f : [a, b] \to \mathbb{R}$ is continuous, and define a new function $g : [a, b] \to \mathbb{R}$ by $g(x) = \mathrm{l}ub\{f(t) \ : \ a \leq t \leq x\}$. Show that g is also continuous!

[Hint: that least upper bound is (always) achieved!]

(*) 44. A function $f : \mathbb{R} \to \mathbb{R}$ is called *periodic* if there is a $P > 0$ so that $f(x+P) = f(x)$ for every $x \in \mathbb{R}$. Show that if f is periodic and continuous, then f is uniformly continuous.

[Hint: you can always insist that your δ's be smaller than some particular positive number (why?).]