Math 106 Calculus 1

Topics for first exam

"Precalculus" = what comes before <u>limits</u>.

Lines and their slopes:

slope= rise over run = (change in y-value)/(corresponding change in x value)

slope-intercept: $y = mx + b$ point-slope: $\frac{y - y_0}{x - x_0} = m$

two-point: $\frac{y - y_0}{x - x_0} = \frac{y_1 - y_0}{x_1 - x_0}$ same slope: lines are parallel (do not meet)

lines are perpendicular: slopes are **negative reciprocals**

Functions: function = rule which assigns to each input **exactly one** output

inputs = domain; outputs = range/image; $f : A \to B$

y=f(x) : 'y equals f of x' : y equals the value assigned to x by the function f

'implied' domain of f: all numbers for which f(x) *makes sense*

Graphs of functions: y=f(x) is an equation; graph the equation!

graph = all pairs (x,f(x)) where x is in the domain of f

function takes only one value at a point; vertical line test

symmetry (for functions)

y-axis: *even* function, f(-x) = f(x) origin: *odd* function, f(-x) = -f(x)

increasing on an interval: if $x > y$, then $f(x) > f(y)$

decreasing on an interval: if $x > y$, then $f(x) < f(y)$

Stretching, shifting, and combinations: start with graph of y=f(x)

shift to right by c; y=f(x-c) shift to left by c; y=f(x+c)

shift down by c; y=f(x)-c shift up by c; y=f(x)+c

y=*a*f(x) ; stretch graph verticcally by factor of a

y=f(ax) ; <u>compress</u> graph horizontally by factor of a

combining functions: combine the outputs of two functions f,g

f+g, f-g, fg, f/g

composition: output of one function is input of the next

f followed by g = g∘f; g∘f(x) = g(f(x)) = g of f **of** x

Inverse functions: Idea: find a function that undoes f

find a function g so that g(f(x)) = x for every x

magic: f undoes g ! Usual notation: g = f^{-1}

Problem: not every function has an inverse.

need g to be a function; f cannot take the same value twice. horizontal line test!

Graph of inverse: if (a,b) on graph of f, then (b,a) is on graph of f^{-1}

graph of f^{-1} is graph of f, reflected across line y=x

Exponential functions.

exponential expressions a^b

Rules: $a^{b+c} = a^b a^c$; $a^{bc} = (a^b)^c$; $(ab)^c = a^c b^c$

Function $f(x) = a^x$; approximate $f(x)$ by f(rational number close to x)

Domain: $\mathbf{R}$; range: $(0, \infty)$; horiz. asymp. $y = 0$

Graphs: $a > 1$: near 0 at left, blows up to right. $0 < a < 1$: reflect in y-axis!
Most natural base: $e = 2.718281829459045.....$
Exponential growth: compound interest
P=initial amount, r=interest rate, compounded n times/year
$A(t) = P \cdot (1 + r/n)^{nt}$ $n \to \infty$, continuous compounding : $A(t) = Pe^{rt}$
Radioactive decay: half-life = k ($A(k) = A(0)/2$) $A(t) = A(0)(1/2)^{t/k}$

Logarithmic functions.
$\log_a x$ = the number you raise a to to get x $\log_a x$ is the inverse of a^x
a = base of the logarithm natural logarithm: $\log_e x = \ln x$
$\log_a(a^x) = x$, all x ; $a^{\log_a x} = x$, all $x > 0$ Domain: all $x > 0$; range: all x
Graph = reflection of graph of a^x across line $y = x$ vertical asymptote: $x = 0$

Properties of logarithms:
logarithms undo exponentials; properties are 'reverse' of exponentials
$\log_a(bc) = \log_a b + \log_a c$; $\log_a(b^c) = c \log_a b$
$(\log_b c)(\log_a b) = \log_a(b^{\log_b c}) = \log_a c$; so $\log_b c = \dfrac{\log_a c}{\log_a b}$ $\log_b c = \dfrac{\ln c}{\ln b}$

Trigonometry.

Degrees and radians: measuring size of an angle
one full circle = 360 degrees
one full circle = 2π radians
radian measure = length of arc in circle of radius 1 swept out by the angle

Trigonometric functions:
Ray making an angle (t) meets circle of radius 1 in a point (x, y)
$x = \cos t$ = cosine of t $y = \sin t$ = sine of t
$\dfrac{1}{x} = \dfrac{1}{\cos t} = \sec t$ = secant of t $\dfrac{1}{y} = \dfrac{1}{\sin t} = \csc t$ = cosecant of t
$\dfrac{y}{y} = \dfrac{\sin t}{\cos t} = \tan t$ = tangent of t $\dfrac{x}{y} = \dfrac{\cos t}{\sin t} = \cot t$ = cotangent of t
Examples:
$\sin(\pi/4) = \cos(\pi/4) = \sqrt{2}/2$ $\sin(\pi/6) = 1/2$; $\cos(\pi/6) = \sqrt{3}/2$
$\sin(\pi/3) = \sqrt{3}/2$; $\cos(\pi/3) = 1/2$ $\sin(\pi/2) = 1$; $\cos(\pi/2) = 0$
$\sin(0) = 0$; $\cos(0) = 1$
Domain of $\sin t$, $\cos t$: all t Range: $[-1, 1]$
point on circle corresp. to $t + 2\pi$ is same as point for t
$\sin(t + 2\pi) = \sin t$; $\cos(t + 2\pi) = \cos t$ $\sin t$ and $\cos t$ are periodic
symmetry:
$\cos t$, $\sec t$ are even functions $\sin t$, $\csc t$, $\tan t$, $\cot t$ are odd functions
$x^2 + y^2 = 1$ (unit circle): $\sin^2 t + \cos^2 t = 1$

Right angle trigonometry:
Right triangle: a=opposite side, b=adjacent side, c=hypotenuse
$\sin(\theta) = a/c$ = (opposite)/(hypotenuse) $\cos(\theta) = b/c$ = (adjacent)/(hypotenuse)
$\tan(\theta) = a/b$ = (opposite)/(adjacent) "SOHCAHTOA"

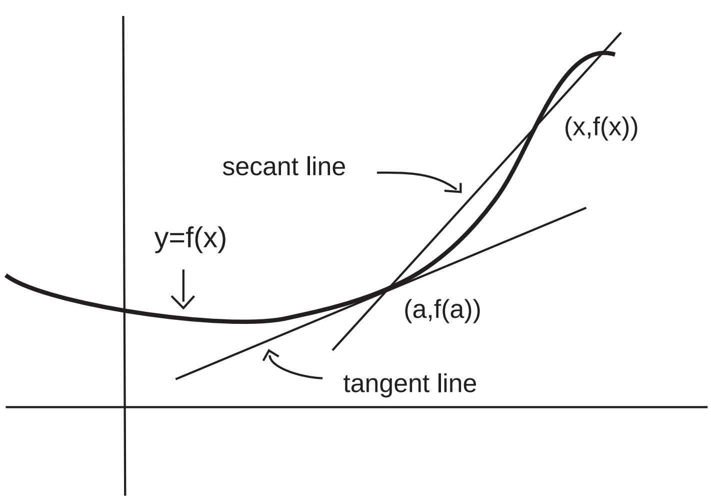
Graphs of sine, cosine:
$\sin(\theta) = y$-value of the points (counter-clockwise) on the unit circle, starting with 0
$\cos(\theta) = x$-value of the points (counter-clockwise) on the unit circle, starting with 1
Graph: note x-intercepts, y-intercept, maximum and minimum; draw a smooth curve
Transformations: $y = a\sin(bx)$
vertical stretch by factor of a; **amplitude** is $|a|$
amplitude = how far trig function wanders from its 'center'
horizontal compression by factor of b; **period** is $2\pi/|b|$
Translations: just like before:
$y = \cos(x - a)$; translation to right by a $\quad$ $y = \cos(x) + a$; translation up by a

Inverse trig functions:
Inverses of trig functions? No! Not one-to-one. Solution: *make* them one-to-one!
$f(x) = \sin x$, $-\pi/2 \le x \le \pi/2$, is one-to-one
inverse is called $\arcsin x$ = angle (between $-\pi/2$ and $\pi/2$) whose sine is x
$\sin(\arcsin x) = x$; $\arcsin(\sin x) = x$ **if** x is between $-\pi/2$ and $\pi/2$
$f(x) = \cos x$, $0 \le x \le \pi$, is one-to-one
inverse is called $\arccos x$ = angle (between 0 and π) whose cosine is x
$\cos(\arccos x) = x$; $\arccos(\cos x) = x$ **if** x is between 0 and π
$f(x) = \tan x$, $-\pi/2 < x < \pi/2$, is one-to-one
inverse is called $\arctan x$ = angle (between $-\pi/2$ and $\pi/2$) whose tangent is x
$\tan(\arctan x) = x$; $\arctan(\tan x) = x$ **if** x is between $-\pi/2$ and $\pi/2$
Graphs: take appropriate piece fo trig function, and flip it across the line $y = x$
$\cos(\arcsin x)$ = (cosine of angle whose **sine** is x) = $\sqrt{1-x^2}$; etc.

Limits and Continuity.

Rates of change and limits:

Limit of a function f at a point a = the value the function 'should' take at the point
= the value that the points 'near' a tell you f should have at a
$\lim_{x\to a} f(x) = L$ means $f(x)$ is close to L when x is close to (but not <u>equal</u> to) a
Idea: slopes of tangent lines



The closer x is to a, the better the slope of the secant line will approximate the slope of the tangent line.

The slope of the tangent line = limit of slopes of the secant lines (through (a,f(a)))

$\lim\limits_{x\to a} f(x) = L$ does <u>not</u> care what $f(a)$ <u>is</u>; it ignores it
$\lim\limits_{x\to a} f(x)$ need not exist! (function can't make up it's mind?)

Rules for finding limits:

If two functions $f(x)$ and $g(x)$ agree (are equal) for every x near a (but maybe not <u>at</u> a), then $\lim\limits_{x\to a} f(x) = \lim\limits_{x\to a} g(x)$

Ex.: $\lim\limits_{x\to 2} \dfrac{x^2-3x+2}{x^2-4} = \lim\limits_{x\to 2} \dfrac{(x-1)(x-2)}{(x+2)(x-2)} = \lim\limits_{x\to 2} \dfrac{x-1}{x+2} = 1/over4$

If $f(x)\to L$ and $g(x)\to M$ as $x\to a$ (and c is a constant), then
$f(x)+g(x)\to L+M$; $f(x)-g(x)\to L-M$; $cf(x)\to cL$;
$f(x)g(x)\to LM$; and $f(x)/g(x)\to L/M$ <u>provided</u> $M\neq 0$

If $f(x)$ is a polynomial, then $\lim\limits_{x\to x_0} f(x) = f(x_0)$

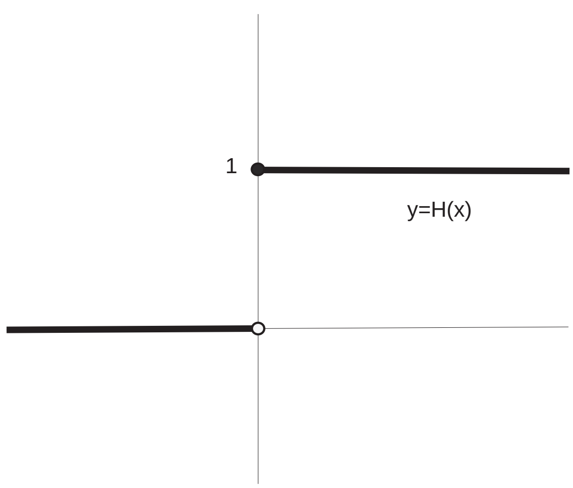
Basic principle: to solve $\lim\limits_{x\to x_0}$, plug in $x = x_0$!

If (and when) you get 0/0 , try something else! (Factor $(x-a)$ out of top and bottom...)
If a function has something like $\sqrt{x}-\sqrt{a}$ in it, try multiplying (top and bottom) with $\sqrt{x}+\sqrt{a}$
(idea: $u=\sqrt{x}, v=\sqrt{a}$, then $x-a = u^2-v^2 = (u-v)(u+v)$.)

Sandwich Theorem: If $f(x)\leq g(x)\leq h(x)$, for all x near a (but not <u>at</u> a), and $\lim\limits_{x\to a} f(x) = \lim\limits_{x\to a} h(x) = L$, then $\lim\limits_{x\to a} g(x) = L$.

One-sided limits:

Motivation: the Heaviside function



The Heaviside function has no limit at 0; it can't make up its mind whether to be 0 or 1. But if we just look to either side of 0, everything is fine; on the left, H(0) `wants' to be 0, while on the right, H(0) `wants' to be 1.

It's because these numbers are different that the limit as we approach 0 does not exist; but the `one-sided' limits DO exist.

Limit from the right: $\lim\limits_{x\to a^+} f(x) = L$ means $f(x)$ is close to L when x is close to, and <u>bigger</u> than, a

Limit from the left: $\lim\limits_{x\to a^-} f(x) = M$ means $f(x)$ is close to M when x is close to, and <u>smaller</u> than, a

$\lim_{x\to a} f(x) = L$ then means $\lim_{x\to a^+} f(x) = \lim_{x\to a^-} f(x) = L$
(i.e., both one-sided limits exist, and are equal)

Limits at infinity / infinite limits:

∞ represents something bigger than any number we can think of.

$\lim_{x\to\infty} f(x) = L$ means $f(x)$ is close of L when x is really large.
$\lim_{x\to-\infty} f(x) = M$ means $f(x)$ is close of M when x is really large and *negative*.

Basic fact: $\lim_{x\to\infty} \frac{1}{x} = \lim_{x\to-\infty} \frac{1}{x} = 0$

More complicated functions: divide by the highest power of x in the denomenator.
$f(x), g(x)$ polynomials, degree of $f = n$, degree of $g = m$

$\lim_{x\to\pm\infty} \frac{f(x)}{g(x)} = 0$ if $n < m$

$\lim_{x\to\pm\infty} \frac{f(x)}{g(x)} =$ (coeff of highest power in f)/(coeff of highest power in g) if $n = m$

$\lim_{x\to\pm\infty} \frac{f(x)}{g(x)} = \pm\infty$ if $n > m$

$\lim_{x\to a} f(x) = \infty$ means $f(x)$ gets really large as x gets close to a
Also have $\lim_{x\to a} f(x) = -\infty$; $\lim_{x\to a^+} f(x) = \infty$; $\lim_{x\to a^-} f(x) = \infty$; etc....

Typically, an infinite limit occurs where the denomenator of $f(x)$ is zero
(although not always)

Asymptotes:

The line $y = a$ is a horizontal asymptote for a function f if
$\lim_{x\to\infty} f(x)$ or $\lim_{x\to-\infty} f(x)$ is equal to a.
I.e., the graph of f gets really close to $y = a$ as $x \to \infty$ or $a \to -\infty$

The line $x = b$ is a vertical asymptote for f if $f \to \pm\infty$ as $x \to b$ from the right or left.
If numerator and denomenator of a rational function have no common roots, then vertical asymptotes = roots of denom.

Continuity:

A function f is <u>continuous</u> (cts) <u>at</u> <u>a</u> if $\lim_{x\to a} f(x) = f(a)$
This means: (1) $\lim_{x\to a} f(x)$ exists ; (2) $f(a)$ exists ; and
(3) they're equal.
Limit theorems say (sum, difference, product, quotient) of cts functions are cts.
Polynomials are continuous at every point;
rational functions are continuous except where denom=0.
Points where a function is not continuous are called <u>discontinuities</u>.
Four flavors:
removable: both one-sided limits are the same
jump: one-sided limts exist, not the same

infinite: one or both one-sided limits is ∞ or $-\infty$
oscillating: one or both one-sided limits DNE

Intermediate Value Theorem:
If $f(x)$ is cts at every point in an interval $[a, b]$, and M is between $f(a)$ and $f(b)$, then there is (at least one) c between a and b so that $f(c) = M$.

Application: finding roots of polynomials!

Tangent lines:

Slope of tangent line = limit of slopes of secant lines; at $(a, f(a)$: $\displaystyle\lim_{x\to a}\frac{f(x)-f(a)}{x-a}$

Notation: call this limit $f'(a)$ = derivative of f at a

Different formulation: $h = x - a$, $x = a + h$

$$f'(a) = \lim_{h\to 0}\frac{f(a+h)-f(a)}{h} = \text{limit of } \textit{difference quotient}$$

If $y = f(x)$ = position at 'time' x, then difference quotient = average velocity; limit = instantaneous velocity.

Derivatives.

The derivative of a function:

derivative = limit of difference quotient (two flavors: $h \to 0$, $x \to a$)
If $f'(a)$ exists, we say f is differentiable at a

Fact: f differentiable (diff'ble) at a, then f cts at a

Using $h \to 0$ notation: replace a with x (= variable), get $f'(x)$ = new function

Or: $f'(x) = \displaystyle\lim_{z\to x}\frac{f(z)-f(x)}{z-x}$

$f'(x)$ = the derivative of f = function whose values are the slopes of the tangent lines to the graph of $y{=}f(x)$. Domain = every point where the limit exists

Notation:

$$f'(x) = \frac{dy}{dx} = \frac{d}{dx}(f(x)) = \frac{df}{dx} = y' = D_x f = Df = (f(x))'$$

Differentiation rules:

$\dfrac{d}{dx}(\text{constant}) = 0$ $\quad\dfrac{d}{dx}(x) = 1$

$(f(x){+}g(x))' = (f(x))'{+}\ (g(x))'$ $\quad(f(x)\text{-}g(x))'{=}\ (f(x))'\text{-}\ (g(x))'$

$(\mathrm{c}f(x))'{=}\ \mathrm{c}(f(x))'$

$(f(x)g(x))'{=}\ (f(x))'g(x){+}\ f(x)(g(x))'$ $\quad\left(\dfrac{f(x)}{g(x)}\right)'{=}\ \dfrac{f'(x)g(x)-f(x)g'(x)}{g^2(x)}$

$(x^n)'{=}\ nx^{n-1}$, for n = ~~natural number~~ ~~integer~~ rational number

$(a^x)'{=}\ K\cdot a^x$, where $K = \dfrac{d}{dx}(a^x)\Big|_{x=0} = \ln a$, so $(a^x)'{=}\ a^x\ln a$

[[$(1/g(x))'{=}\ \text{-}g'(x)/(g(x))^2$]]

Higher derivatives:

$f'(x)$ is 'just' a function, so we can take its derivative!

$$(f'(x))' = f''(x) \qquad (= y'' = \frac{d^2y}{dx^2} = \frac{d^2f}{dx^2}) \qquad = \text{second derivative of } f$$

Keep going! $f'''(x) = $ 3rd derivative, $f^{(n)}(x) = n$th derivative

Rates of change:

Physical interpretation:
$f(t)$= position at time t $\qquad$ $f'(t)$= rate of change of position = velocity
$f''(t)$= rate of change of velocity = acceleration
$|f'(t)|$ = speed
Basic principle: for object to change direction (velocity changes sign),
$f'(t)$= 0 somewhere (IVT!)

Examples:

Free-fall: object falling near earth; $s(t) = s_0 + v_0 t - \frac{g}{2}t^2$

$s_0 = s(0)$ = initial position; v_0 = initial velocity; g= acceleration due to gravity

Economics:
$C(x)$ = cost of making x objects; $R(x)$ = revenue from selling x objects;
$P = R - C$ = profit
$C'(x)$ = marginal cost = cost of making 'one more' object
$R'(x)$ = marginal revenue ; profit is maximized when $P'(x) = 0$;
i.e., $R'(x) = C'(x)$

Derivatives of trigonometric functions:

Basic limit: $\lim_{x\to 0} \frac{\sin x}{x} = 1$; $\qquad$ everything else comes from this! $\qquad$ $\lim_{h\to 0} \frac{1-\cos h}{h} = 0$

Note: this uses radian measure!

Then we get:
$(\sin x)' = \cos x$ $\qquad$ $(\cos x)' = -\sin x$ $\qquad$ $(\tan x)' = \sec^2 x$
$(\cot x)' = -\csc^2 x$ $\qquad$ $(\sec x)' = \sec x \tan x$ $\qquad$ $(\csc x)' = -\csc x \cot x$

The Chain Rule:

Composition $(g \circ f)(x_0) = g(f(x_0))$; $\qquad$ (note: we don't know what $g(x_0)$ is.)
$(g \circ f)'$ ought to have something to do with $g'(x)$ and $f'(x)$
in particular, $(g \circ f)'(x_0)$ should depend on $f'(x_0)$ and $g'(f(x_0))$

Chain Rule: $\qquad$ $(g \circ f)'(x_0) = g'(f(x_0))f'(x_0) = (d(\text{outside}) \text{ eval'd at inside fcn}) \cdot (d(\text{inside}))$

Ex: $((x^3 + x - 1)^4)' = (4(x^3 + 1 - 1)^3)(3x^2 + 1)$

Different notation:
$y = g(f(x)) = g(u)$, where $u = f(x)$, then $\frac{dy}{dx} = \frac{dy}{du}\frac{du}{dx}$