Math 106 Exam 3 Topics

L'Hôpital's Rule

indeterminate forms: limits which 'evaluate' to 0/0 ; e.g. $\lim_{x\to 0} \frac{\sin x}{x}$

LR# 1: If $f(a) = g(a) = 0$, f and g both differentiable near a, then
$$\lim_{x\to a} \frac{f(x)}{g(x)} = \lim_{x\to a} \frac{f'(x)}{g'(x)}$$

Note: we can repeatedly apply L'Hôpital's rule to compute a limit, so long as the condition that top and bottom both tend to 0 holds for the new limit. Once this doesn't hold, L'Hôpital's rule can no longer be applied!

Other indeterminate forms: $\frac{\infty}{\infty}$, $0 \cdot \infty$, $\infty - \infty$, 0^0, 1^∞, ∞^0

LR#2: if $f, g \to \infty$ as $x \to a$, then $\lim_{x\to a} \frac{f(x)}{g(x)} = \lim_{x\to a} \frac{f'(x)}{g'(x)}$

Other cases: try to turn them into 0/0 or ∞/∞. In the $0 \cdot \infty$ case, we can do this by throwing one factor or the other into the denomenator (whichever is more tractable). In the last three cases, do this by taking logs, first.

Parametric Equations

With related rates, we imagined two (related) quantities changing with time, like $x^2+y^2 = 1$ (circle) with $x = x(t)$ and $y = y(t)$. From a different perspective, this describes a point $(x, y) = (x(t), y(t))$ the changes with time, i.e., traces out a curve in the plane.
If we explicitly describe these functions, e.g., $x = \cos t$, $y = \sin t$, we call these *parametric equations* for the curve they tracce out (in this case, the unit circle). Another example: the line through (a, b) and (c, d) has the parametric form $x = a + t(c-a)$, $y = b + t(d-b)$. The same curve, however, can have many different parametrizations! [E.g., the same line is $x = a + t^3(c-a)$, $y = b + t^3(d-b)$.]

Thinking of $x = x(t)$, $y = y(t)$ as motion of a particle through the plane, this particle has a speed $= \sqrt{[x'(t)]^2 + [y'(t)]^2}$ = limit (as $\Delta t \to 0$) of (displacement)/(time).

If the motion described by the parametric equations is 'nice' (i.e., differentiable), the curve has a tangent line at every point. It passes through $(x(t_0), y(t_0))$, and has slope $= y'(t_0)/x'(t_0)$. [This is the basic formula $dy/dx = (dy/dt)/(dx/dt)$, when we think of the curve as 'implicitly' defining y as a function of x.]

Pushing this idea even further, we can compute the second derivative of the implicitly-defined function, as
$$\frac{d^2y}{dx^2} = \frac{d}{dx}\left(\frac{dy}{dx}\right) = \left[\frac{d}{dt}\left(\frac{dy}{dx}\right)\right]/\left[\frac{dx}{dt}\right] = \frac{[y'(t)/x'(t)]'}{x'(t)} = \frac{y''(t)x'(t)-y'(t)x''(t)}{(x'(t))^3}$$

(Should you memorize this? No...)

Integral Calculus

Motivating problem: computing distance traveled, knowing our velocity.

Main idea: if velocity is constant, then distance = velocity·time. When it isn't constant, cut our time interval $[a,b]$ into lots of pieces! (subintervals), and pretend the velocity is constant on each piece (e.g., equal to velocity at start of each subinterval). Then we can approximate distance traveled as a sum the distances traveled over each time subinterval, which we estimate to be (velocity at start)·(length of subinterval). We expect that if we use smaller subintervals (= lots more subintervals), our estimates become better. So in the limit(!) as the lengths go to zero, we recover the distance traveled.

The other main idea: If $s(t)$ represents how far we have traveled at time t, then $s'(t) = v(t)$ is the velocity. But any two functions with the same derivative differ by a constant (bt the Mean Value Theorem). So if we know $v(t)$ as a function, and we know how to find a function $f(t)$ with $f(a) = 0$ and $f'(t) = v(t)$, then $f(t)$ must be $s(t)$, so the distance traveled is $f(b)$. Playing these two ideas off of one another will lead us to a remarkably powerful tool.

Another look at these sums: imagine cutting $[a,b]$ into pieces by $a = x_0 < x_1 < x_2 < \dots x_{n-1} < x_n = b$, then each number is $v(x_{k-1}) \cdot (x_k - x_{k-1}) = v(x_{k-1})\Delta x_k$. If we think of this as (height)$\times$(width), then these are sums of area of rectangles, which fit around the graph of $y = v(x)$. Taken together, these rectangles are trying to approximate the region lying under the graph of $y = v(x)$ between $x = a$ and $x = b$. This leads us to:

The distance traveled by an object whose velocity is given by the function $y = v(t)$ from $t = a$ to $t = b$ is equal to the area under the graph of $y = v(t)$ between $t = a$ and $t = b$.

Sums and Sigma Notation.

Idea: a lot of things can estimated by adding up alot of tiny pieces.

Sigma notation: $\displaystyle\sum_{i=1}^{n} a_i = a_1 + \cdots a_n$; just add the numbers up

Formal properties: $\displaystyle\sum_{i=1}^{n} ka_i = k\sum_{i=1}^{n} a_i$ $\qquad\qquad$ $\displaystyle\sum_{i=1}^{n}(a_i \pm b_i) = \sum_{i=1}^{n} a_i \pm \sum_{i=1}^{n} b_i$

Some things worth adding up:

length of a curve: approximate curve by a collection of straight line segments

length of curve $\approx \sum$(length of line segments)

distance travelled = (average velocity)(time of travel)

over short periods of time, avg. vel. $\approx$ instantaneous vel.

so distance travelled $\approx \sum$(inst. vel.)(short time intervals)

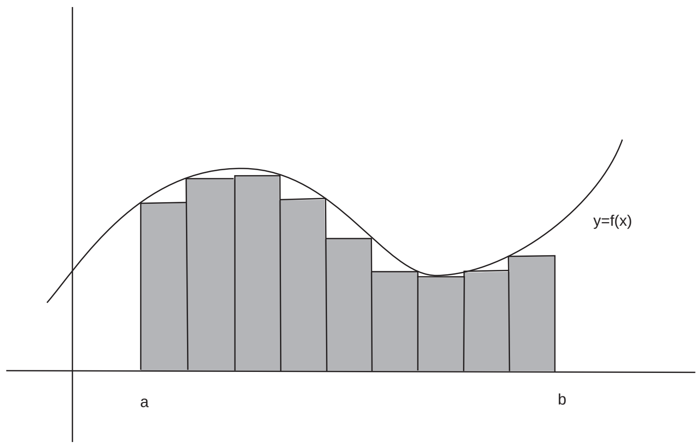
Average value of a function:

Average of n numbers: add the numbers, divide by n . For a function, add up lots of values of f, divide by number of values.

avg. value of $f \approx \displaystyle\frac{1}{n}\sum_{i=1}^{n} f(c_i)$

Area and Definite Integrals.

Probably the most important thing to approximate by sums: area under a curve.
Idea: approximate region b/w curve and x-axis by things whose areas we can easily calculate: **rectangles**!



Area between graph and x-axis $\approx \sum$ (areas of the rectangles) $=\sum_{i=1}^{n} f(c_i)\Delta x_i$

where c_i is chosen inside of the i-th interval that we cut $[a,b]$ up into. This is a Riemann sum for the function f on the interval $[a,b]$.)

We define the area to be the limit of these sums as the lengths of the subintervals gets small (so the number of rectangles goes to ∞, and call this the *definite integral* of f from a to b:

$$\int_a^b f(x)\,dx \;=\; \lim_{n\to\infty} \sum_{i=1}^{n} f(c_i)\Delta x_i$$

More precisely, we can at all Riemann sums, and look at what happens when the length Δx_i of the largest subinterval (call it Δ) gets small. If the Riemann sums all approximate some number I when Δ is small enough, then we call I the definite integral of f from a to b. But when do such limits exist?

Theorem If f is continuous on the interval $[a,b]$, then $\int_a^b f(x)\,dx$ exists.

(i.e., the area under the graph is approximated by rectangles.)

But this isn't how we want to compute these integrals! Limits of sums is very cumbersome. Instead, we try to be more systematic.

Properties of definite integrals:

First note: the sum used to define a definite integral doesn't need to have $f(x) \geq 0$; the limit still makes sense. When f is bigger than 0, we interpret the integral as area under the graph.

Basic properties of definite integrals:

$\int_a^a f(x)\ dx = 0$ $\qquad$ $\int_b^a f(x)\ dx = -\int_a^b f(x)\ dx$

$\int_a^b kf(x)\ dx = k\int_a^b f(x)\ dx$ $\qquad$ $\int_a^b f(x) \pm g(x)\ dx = \int_a^b f(x)\ dx \pm \int_a^b g(x)\ dx$

$\int_a^b f(x)\ dx + \int_b^c f(x)\ dx = \int_a^c f(x)\ dx$

If $m \le f(x) \le M$ for all x in $[a,b]$, then $m(b-a) \le \int_a^b f(x)\ dx \le M(b-a)$

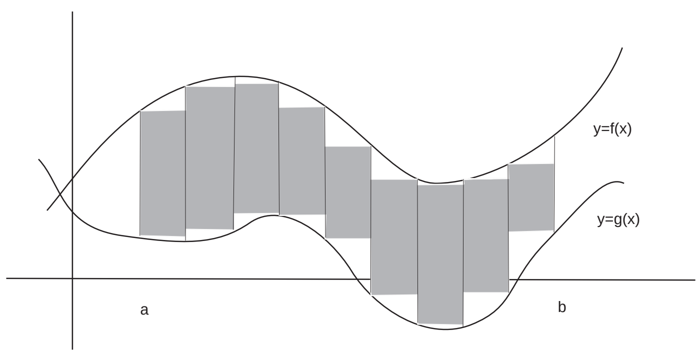
More generally, if $f(x) \le g(x)$ for all x in $[a,b]$, then $\int_a^b f(x)\ dx \le \int_a^b g(x)\ dx$

Average value of f : formalize our old idea! $\quad \text{avg}(f) = \frac{1}{b-a}\int_a^b f(x)\ dx$

Mean Value Theorem for integrals: If f is continuous in $[a,b]$, then there is a c in $[a,b]$ so that $f(c) = \frac{1}{b-a}\int_a^b f(x)\ dx$

Area between curves

Region between two curves; approximate by rectangles



Area $= \int_{left}^{right} (top) - (bottom)\ dx = \int_a^b f(x) - g(x)\ dx$

Integrate dy: Area $= \int_{bottom}^{top} (right) - (left)\ dy$

ssk

If what the function at top/bottom <u>is</u> changes, cut the interval into pieces, and

use $\int_a^b = \int_a^c + \int_c^b$

Sometimes to calculate area between $f(x)$ and $g(x)$, need to first figure out limits of integration; solve $f(x) = g(x)$, then decide whicxh one is bigger in between each pair of solutions.

Antiderivatives.

Integral calculus is all about finding areas of things, e.g. the area between the graph of a function f and the x-axis. This will, in the end, involve finding a function F whose *derivative* is f.

F is an *antiderivative* (or (indefinite) *integral*) of f if $F'(x) = f(x)$.
Notation: $F(x) = \int f(x)\ \mathrm{d}x$; it <u>means</u> $F'(x)=f(x)$; "the integral of f of x dee x"

Every differentiation formula we have encountered can be turned into an antidifferentiation formula; if g is the derivative of f, then f is an antiderivative of g. Two functions with the same derivative (on an interval) differ by a constant, so <u>all</u> antiderivatives of a function can be found by finding one of them, and then adding an arbitrary constant C.

Basic list:

$\int x^n\ \mathrm{d}x = \dfrac{x^{n+1}}{n+1} + C$ (provided $n \neq -1$) $\qquad \int 1/x\ \mathrm{d}x = \ln|x| + C$

$\int \sin x\ \mathrm{d}x = -\cos x + C \qquad \int \cos x\ \mathrm{d}x = \sin x + C$

$\int \sin(kx)\ \mathrm{d}x = \dfrac{-\cos(kx)}{k} + C \qquad \int \cos(kx)\ \mathrm{d}x = \dfrac{\sin(kx)}{k} + C$

$\int \sec^2 x\ \mathrm{d}x = \tan x + C \qquad \int \csc^2 x\ \mathrm{d}x = -\cot x + C$

$\int \sec x \tan x\ \mathrm{d}x = \sec x + C \qquad \int \csc x \cot x\ \mathrm{d}x = -\csc x + C$

$\int e^x\ \mathrm{d}x = e^x + C \qquad \int a^x\ dx = \frac{a^x}{\ln a} + C$

Most differentiation rules can be turned into integration rules (although some are harder than others; some we will wait awhile to discover).

Basic integration rules: sum and constant multiple rules are straighforward to reverse: for k=constant,

$\int k \cdot f(x)\ \mathrm{d}x = k \int f(x)\ \mathrm{d}x \qquad \int (f(x) \pm g(x)\ \mathrm{d}x = \int f(x)\ \mathrm{d}x \pm \int g(x)\ \mathrm{d}x$

Differential Equations

A *differential equation* refers to an equation, like $y' = 3y$, which involves an <u>unknown</u> function y and some of its derivatives. The goal is to determine what function(s) satisfy the equation. For our purposes we will focus on equations of the form $y' = f(x)$ for some <u>known</u> function f; such an equation essentially asks us to find an antiderivative of f.
As we have seen, these equations generally have a <u>family</u> of solutions, since there is a "constant of integration" involved with antidifferentiation. We can specify that a single solution is wanted by imposing additional conditions; typically this means insisting that at some $x = a$ we have $y = b$. For example, the differential equation $y' = e^{2x}$ has solution(s) $y = \frac{1}{2}e^{2x} + C$ (as you can verify), insisting that $y(0) = 0$ specifies the unique solution $y = \frac{1}{2}e^{2x} - \frac{1}{2}$.
We can apply these ideas to ballistic motion, where the basic equation that governs motion $y = s(t)$ near the Earth's surface is $s''(t) = -9.8$ m/sec^2 (measured in meters) or $s''(t) = -32$ (measured in feet). We can solve this equation to find $s'(t) = -9.8t + C$; solving for C gives $C = s'(0) = v_0 =$ initial velocity. Then solving $s'(t) = -9.8t + v_0$ gives $s(t) = -4.9t^2 + v_0 t + C$; again solving for C gives $C = s(0) = s_0$ =initial position. So that equation for ballistic motion (height above the Earth's surface under the influence of gravity) is $s(t) = -4.9t^2 + v_0 t + s_0$ (measured in meters).

The fundamental theorems of calculus.

Formally, $\displaystyle\int_a^b f(x)\,dx$ depends on a and b. Make this explicit:

$\displaystyle\int_a^x f(t)\,dt = F(x)$ is a function of x.

$F(x) =$ the area under the graph of f, from a to x.

Fund. Thm. of Calc (# 1): If f is continuous, then $F'(x) = f(x)$ (F is an antiderivative of f !)

Since any two antiderivatives differ by a constant, and $F(b) = \displaystyle\int_a^b f(t)\,dt$, we get

Fund. Thm. of Calc (# 2): If f is continuous, and F is an antiderivative of f, then $\displaystyle\int_a^b f(x)\,dx = F(b) - F(a) = F(x)\Big|_a^b$

Ex: $\displaystyle\int_0^\pi \sin x\,dx = (-\cos\pi) - (-\cos 0) = 2$

FTC # 2 makes finding antiderivatives very important! FTC # 1 gives a method for building antiderivatives:

$F(x) = \displaystyle\int_a^x \sqrt{\sin t}\,dt$ is an antiderivative of $f(x) = \sqrt{\sin x}$

$G(x) = \displaystyle\int_{x^2}^{x^3} \sqrt{1+t^2}\,dt = F(x^3) - F(x^2)$, where $F'(x) = \sqrt{1+x^2}$,

so $G'(x) = F'(x^3)(3x^2) - F'(x^2)(2x) = (3x^2)\sqrt{1+(x^3)^2} - (2x)\sqrt{1+(x^2)^2}$.

The following topic is <u>NOT</u> on Exam 3; it will be covered on the final exam.

Integration by substitution.

The rules we have tell us that the sums, differences, and constant multiples of functions whose integrals we can handle we can <u>also</u> handle. Further rules allow us to relate the antiderivatives of functions to the antiderivatives of "simpler" functions.

The idea: reverse the chain rule!

If $g(x) = u$, then $\frac{d}{dx}f(g(x))=\frac{d}{dx}f(u) = f'(u)\ \frac{du}{dx}$

so $\int f'(u)\ \frac{du}{dx}\ dx = \int f'(u)\ du = f(u) + c$

So: faced with $\int f(g(x))g'(x)\ dx$, set $u = g(x)$, then $du = g'(x)\ dx$, so $\int f(g(x))g'(x)\ dx = \int f(u)\ du$, <u>where</u> $u = g(x)$

Example: $\int x(x+2-3)^4\ dx$; set $u = x^2 - 3$, so du=$2x\ dx$. Then

$\int x(x+2-3)^4\ dx = \frac{1}{2}\int (x+2-3)^4 2x\ dx = \frac{1}{2}\int u^4\ du\ |_{u=x^2-3} =$

$\frac{1}{2}\frac{u^5}{5} + c\ |_{u=x^2-3} = \frac{(x^2-3)^5}{10} + c$

$\displaystyle\int \tan x\ dx = \int \frac{\sec x \tan x}{\sec x}\ dx$; using the substitution $u = \sec x$ we get $\displaystyle\int \tan x\ dx = \ln|\sec x| + C = -\ln|\cos x| + C$. Similarly, $\int \csc x\ dx = \ln|\sin x| + c$.

The three most important points:

1. Make sure that you calculate (and then set aside) your du before doing step 2!
2. Make sure everything gets changed from x's to u's
3. <u>**Don't**</u> push x's through the integral sign! They're <u>not</u> constants!

We can use u-substitution directly with a definite integral, provided we remember that $\displaystyle\int_a^b f(x)\ dx$ <u>really</u> means $\displaystyle\int_{x=a}^{x=b} f(x)\ dx$, and we remember to change <u>all</u> of the x's to u's!

Ex: $\displaystyle\int_1^2 x(1+x^2)^6\ dx$; set $u = 1+x^2$, $du = 2x\ dx$. when $x = 1$, $u = 2$; when $x = 2$, $u = 5$;

so $\displaystyle\int_1^2 x(1+x^2)^6\ dx = \frac{1}{2}\int_2^5 u^6\ du = ...$