Math 325 Topics Sheet for Exam 1

The Real Line: Nearly everything we will do comes down to understanding the properties of the real line $\mathbb{R}$.

$\mathbb{R}$ = a *complete ordered field*

Field: we have addition $+$ and multiplication $\cdot$, so that
(A1) addition exists: if $x, y \in \mathbb{R}$ then $x + y \in \mathbb{R}$
(A2) commutativity: $x + y = y + x$ for every $x, y \in \mathbb{R}$
(A3) associativity: $x + (y + z) = (x + y) + z$ for every $x, y, z \in \mathbb{R}$
(A4) zero exists: there is $0 \in \mathbb{R}$ so that $x + 0 = x$ for every $x \in \mathbb{R}$
(A5) additive inverses exist: for every $x \in \mathbb{R}$ there is a $(-x) \in \mathbb{R}$ with $x + (-x) = 0$
(M1) multiplication exists: if $x, y \in \mathbb{R}$ then $x \cdot y \in \mathbb{R}$
(M2) commutativity: $x \cdot y = y \cdot x$ for every $x, y \in \mathbb{R}$
(M3) associativity: $x \cdot (y \cdot z) = (x \cdot y) \cdot z$ for every $x, y, z \in \mathbb{R}$
(M4) one exists: there is $1 \in \mathbb{R}$ so that $x \cdot 1 = x$ for every $x \in \mathbb{R}$
(M5) multiplicative inverses exist:
for every $x \in \mathbb{R}$ with $x \neq 0$ there is a $x^{-1} \in \mathbb{R}$ with $x \cdot x^{-1} = 1$
(D) distributivity: for every $x, y, z \in \mathbb{R}$ we have $x(y + z) = xy + xz$

Ordered: there is a collection $\mathcal{P}$ = the *positive numbers* so that
(P1) closed under addition: if $x, y \in \mathcal{P}$ then $x + y \in \mathcal{P}$
(P2) closed under multiplication: if $x, y \in \mathcal{P}$ then $x \cdot y \in \mathcal{P}$
(P3) trichotomy:
for every $x \in \mathbb{R}$ *exactly one* of the following is true: $x \in \mathcal{P}$, or $-x \in \mathcal{P}$, or $x = 0$.

We then defined the ordering $x < y$ to mean that $y - x \in \mathcal{P}$. (So $x > 0$ means $x \in \mathcal{P}$.)
Basic properties:
trichotomy! for any $x, y \in \mathbb{R}$ exactly one of $x < y$, $x = y$, or $x > y$ is true.
transitivity: $x < y$ and $y < z$ implies that $x < z$
$x < y$ implies that $x + z < y + z$ for any $z \in \mathbb{R}$
$x < y$ and $z > 0$ implies that $xz < yz$ (because $yz - xz = (y - x)z$ with $y - x, z \in \mathcal{P}$)
weak inequality: $a \leq b$ means $a < b$ or $a = b$; similar properties hold!

From these basic properties we can recover many familiar properties we have seen before; for example
$(-x)y = -(xy)$ $\quad$ $(-x)(-y) = xy$ $\quad$ $-(-x) = x$ $\quad$ $(x^{-1})^{-1} = x$
$x < 0$ and $y > 0$ implies $xy < 0$ $\quad$, $\quad$ $z < 0$ and $w < 0$ implies $zw > 0$
the additive inverse of a number is unique (i.e., if $x + y = 0 = x + z$ then $y = z$)

Completeness: From the natural (= counting) numbers $\mathbb{N}$ we get the integers $\mathbb{Z}$ (by taking additive inverses) and then the rationals $\mathbb{Q}$ (by taking multiplicative inverses). But to get the reals $\mathbb{R}$ we need to step beyond the properties above.

A set $A \subseteq \mathbb{R}$ is *bounded (bdd) from above* if there is a $M \in \mathbb{R}$ so that $x \leq M$ for every $x \in A$.
A *least upper bound* λ is an upper bound for A so that no *smaller* number is an upper bound. In symbols: $x \leq \lambda$ for every $x \in A$ and if $\mu < \lambda$ then there is an $x \in A$ with $\mu < x$

[Equivalently: λ is an upper bd for A and if ν is *also* an upper bound for A then $\lambda \leq \nu$.]

Completeness Axiom: Every *non-empty* set $A \subseteq \mathbb{R}$ that is bdd from above *has* a least upper bound.

Least upper bound of A is unique! $\lambda = \sup(A)$

Application: If $x, y \in \mathbb{R}$ and $y - x > 1$, then there is an $n \in \mathbb{Z}$ with $x \leq n < y$.

Application: $A = \{x \in \mathbb{R} \ : \ x^2 < 2\}$ is non-empty and bdd above: $\lambda = \sup(A)$. Then we showed: $\lambda^2 = 2$ (!) So $\lambda =$ what we would call $\sqrt{2}$

Rational Roots Theorem: If $p(x) = a_0x^n + \cdots a_{n-1}x + a_n$ is a polynomial with integer coefficients $a_i \in \mathbb{Z}$ for all i), and if $r = \alpha/\beta$ is a rational root of p ($p(\alpha/\beta) = 0$ where α and β have no factors in common, *then* α evenly divides a_n and β evenly divides a_0.

Since $\sqrt{2}$ is a root of $p(x) = x^2 - 2$, which by the rational roots theorem has no rational roots, $\sqrt{2} \notin \mathbb{Q}$. (!) By the same reasoning, if $n \in \mathbb{N}$ and $\sqrt{n} \in \mathbb{Q}$ then $\sqrt{n} \in \mathbb{N}$.

Archimedean Property: If $A \subseteq \mathbb{Z}$ is bounded above, then it has a *largest element.* That is, there is an $\lambda \in A$ so that $z \leq \lambda$ for every $z \in A$. (In general for a set A whose supremum λ lies in A, we call it the *maximum* of A.)

Application: the *Principle of Mathematical Induction:* If $A \subseteq \mathbb{N}$ is a set satisfying
(1) $n_0 \in A$, and (2) if $n \geq n_0$ and $n \in A$ then $n + 1 \in A$, *then* $\{n \in \mathbb{N} \ : \ n \geq n_0\} \subseteq A$.

PMI as it is usually stated: If $P(n)$ is a statement about the integer n so that
(1) $P(n_0)$ is true, and
(2) if $n \geq n_0$ and $P(n)$ is true then $P(n+1)$ must also be true,
then $P(n)$ is true for every integer $n \geq n_0$.

Sample applications:

For every integer $n \geq 1$ we have $\displaystyle\sum_{k=1}^{n} k \ = \ \frac{n(n+1)}{2}$

For every integer $n \geq 1$ we have $\displaystyle\sum_{k=1}^{n} k^2 \ = \ \frac{n(n+1)(2n+1)}{6}$

For every $n \geq 5$ we have $2^n > n^2$

Fibonacci sequence: $a_0 = a_1 = 1$ and for $n \geq 1$ we have $a_{n+1} = a_n + a_{n-1}$. Then for every $n \geq$ (something) we have $a_n \geq \left(\frac{3}{2}\right)^n$, and for every $n \geq 12$ we have $a_n \geq n^2$.

[For all of these the method by which we established the inductive step is probably more important as we go forward than the actual result!]

More Archimedean property consequences:
If $\epsilon > 0$ then there is an $n \in \mathbb{N}$ so that $n\epsilon > 0$.
[Add a positive number to itself enough times and you will get a big number!]
Rationals are everywhere: If $x, y \in \mathbb{R}$ with $x < y$ then there is an $r \in \mathbb{Q}$ so that $x < r < y$.

Sequences and convergence: Start with *distance*:
Absolute value: $|x| = x$ if $x \geq 0$, otherwise it is $-x$. So $|x| \geq 0$ for all x.
"Triangle inequality": $|x+y| \leq |x| + |y|$ for every $x, y \in \mathbb{R}$.
[Useful 'opposite' consquence: $|x-y| \geq |x| - |y|$
(useful for showing that $|x-y|$ is <u>not</u> small!]

$|x-y|$ = the distance between x and y. Triangle inequality: $|x-z| \leq |x-y| + |y-z|$

Sequences: A sequence is essentially just a function $f : \mathbb{N} \to \mathbb{R}$. We write $f(n) = a_n$

Examples: the Fibonnaci sequence! $a_0 = a_1 = 1$, $a_2 = 2$, $a_3 = 3$, $a_4 = 5$, $a_5 = 8$, $a_6 = 13$ ...

$a_n = 1 + (-1)^n/n$, $b_n = \dfrac{n^2 - n + 13}{3n^2 + 2n - 13}$

$a_n = \displaystyle\sum_{k=1}^{n} \frac{1}{k^2}$

Convergence: What happens to a sequence as n gets large? E.g., for $b_n = \dfrac{n^2 - n + 13}{3n^2 + 2n - 13}$, observation/intuition/calculus suggests that for large n b_n is approximately 1/3. Formalizing this notion leads us to:

$\lim_{n \to \infty} a_n = L$ <u>means</u> that a_n is <u>close</u> <u>to</u> L so long as n is <u>large</u>. More precisely, a_n is as close as we want it to be to L, so long as n is <u>large</u> <u>enough</u>. This leads us to

$\lim_{n \to \infty} a_n = L$ <u>means</u> for every $\epsilon > 0$ [think: small!] there is an $N \in \mathbb{N}$ [N <u>depends</u> on ϵ (!)] so that $n \geq N$ implies that $|a_n - L| < \epsilon$. If this limit <u>exists</u> (i.e., there is an L that the sequence converges to) we say that the sequence is *convergent*. If there is no number that the sequence converges to, we say that the sequence is *divergent*. [Shorthand: we write $a_n \to L$.]

The key here is that we wish to show that $|a_n - L|$ is small; we typically do this by comparing $|a_n - L|$ to other things that we <u>know</u> are small. This, in turn, we usually do by altering $|a_n - L|$, making it larger (but not too large!), until we end up with something we can show is small (so long as n is large enough!). To help us do this we have some basic limit results:

If $a_n \to L$ and $b_b \to M$, and $k \in \mathbb{R}$ is a konstant, then:
$ka_n \to kL$
$a_n + b_n \to L + M$
$a_n - b_n \to L - M$
$a_n b_n \to LM$
$a_n / b_n \to L/M$ (<u>so</u> <u>long</u> <u>as</u> $b_n, M \neq 0$)

Other properties we learned along the way:

If $(a_n)_{n=1}^{\infty}$ is convergent, then the set $A = \{a_n \ : \ n \in \mathbb{N}\}$ is bounded.
Limits are unique! If $a_n \to L$ and $a_n \to M$, then $L = M$.
$\dfrac{1}{n} \to 0$
If $a_n \geq a$ for every n (eventually!) and $a_n \to L$, then $L \geq a$

If $a_n \geq b_n$ for every n and $a_n \to L$, $b_n \to M$, then $L \geq M$
If $a_n \to L$ and $L \neq 0$, then eventually $|a_n| > \dfrac{|L|}{2}$
If $a_n \to L$ with $a_n, L \geq 0$ for all n, then $\sqrt{a_n} \to \sqrt{L}$
If If $a_n \to L$ and $L > 0$, then eventually $a_n > \dfrac{L}{2} > 0$

Showing convergence without knowing the limit: monotonicity

a_n is *monotone increasing* if $a_{n+1} \geq a_n$ for every n
b_n is *monotone decreasing* if $b_{n+1} \leq b_n$ for every n
If it is one or the other, we say the sequence is *monotone* (or *monotonic*).
A monotone increasing sequence that is bounded above (i.e., for some $M \in \mathbb{R}$ we have $a_n \leq M$ for every $n \in \mathbb{N}$) *converges* it limit is $\sup\{a_n \ : \ n \in \mathbb{N}\}$
[A monotone decreasing sequence that is bounded below also converges!]

Examples:

$a_n = \displaystyle\sum_{k=1}^{n} \frac{1}{k^2} \leq 1 + \sum_{k=2}^{n} \frac{1}{k(k-1)} = 1 + \frac{n-1}{n} \leq 2$ (where the last equality is established by induction!), so $a_n \to L$ for some L [Fact: $L = \pi^2/6$ ...]

$a_1 = 7$ and $a_{n+1} = \sqrt{5 + a_n}$ is monotone decreasing and bounded below (by 0), so converges. $a_n \to L$ satisfies $L = \sqrt{5 + L}$ (by uniqueness!), allowing us to compute L !

An 'intrinsic' characterization of convergence: Cauchy sequences
If $a_n \to L$, then the terms of the sequence eventually get close to one another: if $n \geq N$ implies $|a_n - L < \epsilon$ then $n, m \geq N$ implies

$$|a_n - a_m| = |(a_n - L) + (L - a_m)| \leq |a_n - L| + |L - a_m| = |a_n - L| + |a_m - L| < \epsilon + \epsilon = 2\epsilon$$

This leads to the notion of a *Cauchy sequence*:
$(a_n)_{n=1}^{\infty}$ is Cauchy if for every $\epsilon > 0$ there is an N so that $n, m \geq N$ implies $|a_n - a_m| < \epsilon$.

Convergent sequences are Cauchy.
More surprisingly, every Cauchy sequence is convergent! I.e., $(a_n)_{n=1}^{\infty}$ Cauchy implies that there is an L so that $a_n \to L$.

Finding L: Cauchy sequences are bounded. Then setting $b_n = \sup\{a_k \ : \ k \geq n\}$, the sequence b_n is monotone *decreasing*, and bounded below (by the lower bound of the entire sequence a_n), so b_n converges to some number L. <u>Then</u> $a_n \to L$, because for for any $\epsilon > 0$ there is an N so that for $n \geq N$ we have

$$|a_n - a_N| < \epsilon/3 \text{ and } |a_N - b_N| < \epsilon/3 \text{ and } |b_N - L| < \epsilon/3;$$

adding together gives $|a_n - L| < \epsilon$.