Math 325 Final Exam

Due Wednesday, May 3, at 5:30pm (the end of our scheduled final exam period).

Instructions: This exam is a self-timed exam. You should limit yourself to five (5) hours of active work on the exam. You may partition that time in any way that best fits your schedule. In preparing your answers, you may consult our textbook, your course notes, your solutions to the problem sets, the instructor's solutions to the problem sets, and the topics sheets prepared by the instructor. You should not consult any other resource (to the extent that your other studies allow) to aid you in formulating your solutions, nor discuss the exam problems with anyone other than your instructor, except on trivial matters, until after 5:30pm on May 3. Should you have any questions on the meaning or scope of the problems, or any other questions about the problems and their solutions, feel free to discuss them with your instructor at any time that he is available. Your solutions may be handed in to your instructor in person, in his mailbox, or under his office door, whichever is most convenient, at any time up to the scheduled completion time above. **All (five) problem numbers have equal weight.**

1. If $\{a_n\}_{n=1}^{\infty}$ is a convergent sequence, with $a_n \to L$ as $n \to \infty$, then show that the sequence b_n defined by

$$b_n = \frac{a_n + \cdots + a_{2n}}{n} = \frac{1}{n}\sum_{k=n}^{2n} a_k$$

also has $b_n \to L$ as $n \to \infty$.

2. Show, using the Intermediate Value Theorem, that there is a positive real number α which satifies $\alpha^2 = \alpha + \sqrt{3}$, and show that your solution cannot satisfy $\alpha \in \mathbb{Q}$.

3. Use L'Hôpital's Rule to find an $a \in \mathbb{R}$ so that the function

$$f(x) = \begin{cases} \frac{\sin x - \tan x}{x^2} & \text{if } x \neq 0 \\ a & \text{if } x = 0 \end{cases}$$

is continuous as $x = 0$, and compute the derivative $f'(0)$ of f at $x = 0$.

4. Show that if f and g are integrable on $[a, b]$ and, for every partition

$$P = \{a = x_0 < x_1 < \cdots < x_n = b\}$$

of $[a, b]$, there are sample points $x_{i-1} \le c_i \le x_i$ and $x_{i-1} \le d_i \le x_i$ so that

$R(f, P, \{c_i\}) \le R(g, P, \{d_i\})$, then $\displaystyle\int_a^b f(x)\, dx \le \int_a^b g(x)\, dx$.

5. Show that if $f : [a, b] \to \mathbb{R}$ is a function, then f is integrable on $[a, b]$ if and only if, for any $\epsilon > 0$, there is a $\delta > 0$ so that for every partition P of $[a, b]$ with $||P|| < \delta$ and sample points $\{c_i\}$,$\{d_i\}$ for P we have $\displaystyle\sum_{i=1}^{n} |f(c_i) - f(d_i)| \cdot |x_i - x_{i-1}| < \epsilon$.

[Hint: show that this implies one of our known criteria, and another of our criteria implies this!]