Math 325 Topics Sheet for Exam 1

Throughout, we rely on set notation to make our ideas precise. Sets are typically described as the collection of all objects from a specific 'universe' that meet certain specific conditions:

E.g., $S = \{x \in \mathbb{Q} \mid x^2 < 2\}$ = the rational numbers (universe!) whose square is less than 2 (condition!).

$A \subseteq B$ means that every element of A is also an element of B; $A = B$ is typically established by showing that $A \subseteq B$ and $B \subseteq A$.

$A \cup B = \{x \mid x \in A \ textrmor\ x \in B\}$; $A \cap B = \{x \mid x \in A \ textrmand\ x \in B\}$; $A^c = \{x \mid x \notin A\}$

Functions: a function $f : A \to B$ is a rule which assigns to each $x \in A$ (the *domain*) exactly one element $f(x) \in B$ (the *codomain*).

We will mostly focus on functions with $A \subseteq \mathbb{R}$ and $B \subseteq \mathbb{R}$. Functions come in different flavors:
increasing: if $x > y$ then $f(x) > f(y)$. non-decreasing: $x \geq y$ implies $f(x) \geq f(y)$.
decreasing: if $x > y$ then $f(x) < f(y)$. non-increasing: $x \geq y$ implies $f(x) \leq f(y)$.
One-to-one: if $f(x) = f(y)$, then $x = y$. Alternate form: if $x \neq y$ then $f(x) \neq f(y)$. Third form!: for any $y \in B$, there is at most one $x \in A$ with $f(x) = y$.
Onto: for every $y \in B$, there is at least one $x \in A$ with $f(x) = y$.
One-to-one and onto = a one-to-one correspondence (or *bijection*).

Note: one-to-one and onto have a lot to do with what the domain and codomain of the funciton f are!

Composition: $f : A \to B$, $g : B \to C$, then $g \circ f : A \to C$ is $(g \circ f)(x) = g(f(x))$

If $g \circ f$ is onto, then g is onto! If $g \circ f$ is one-to-one, then f is one-to-one!

The Real Line: Nearly everything we will do comes down to understanding the properties of the real line $\mathbb{R}$.

$\mathbb{R}$ = a *complete ordered field*

Field: we have addition + and multiplication $\cdot$, so that
(A1) addition exists: if $x, y \in \mathbb{R}$ then $x + y \in \mathbb{R}$
(A2) commutativity: $x + y = y + x$ for every $x, y \in \mathbb{R}$
(A3) associativity: $x + (y + z) = (x + y) + z$ for every $x, y, z \in \mathbb{R}$
(A4) zero exists: there is $0 \in \mathbb{R}$ so that $x + 0 = x$ for every $x \in \mathbb{R}$
(A5) additive inverses exist: for every $x \in \mathbb{R}$ there is a $(-x) \in \mathbb{R}$ with $x + (-x) = 0$
(M1) multiplication exists: if $x, y \in \mathbb{R}$ then $x \cdot y \in \mathbb{R}$
(M2) commutativity: $x \cdot y = y \cdot x$ for every $x, y \in \mathbb{R}$
(M3) associativity: $x \cdot (y \cdot z) = (x \cdot y) \cdot z$ for every $x, y, z \in \mathbb{R}$
(M4) one exists: there is $1 \in \mathbb{R}$ so that $x \cdot 1 = x$ for every $x \in \mathbb{R}$
(M5) multiplicative inverses exist:
for every $x \in \mathbb{R}$ with $x \neq 0$ there is a $x^{-1} \in \mathbb{R}$ with $x \cdot x^{-1} = 1$
(D) distributivity: for every $x, y, z \in \mathbb{R}$ we have $x(y + z) = xy + xz$

Ordered: there is a collection $\mathcal{P}$ = the *positive numbers* so that
(P1) closed under addition: if $x, y \in \mathcal{P}$ then $x + y \in \mathcal{P}$
(P2) closed under multiplication: if $x, y \in \mathcal{P}$ then $x \cdot y \in \mathcal{P}$
(P3) trichotomy:

for every $x \in \mathbb{R}$ *exactly one* of the following is true: $x \in \mathcal{P}$, or $-x \in \mathcal{P}$, or $x = 0$.

We then defined the ordering $x < y$ to mean that $y - x \in \mathcal{P}$. (So $x > 0$ means $x \in \mathcal{P}$.)
Basic properties:
trichotomy! for any $x, y \in \mathbb{R}$ exactly one of $x < y$, $x = y$, or $x > y$ is true.
transitivity: $x < y$ and $y < z$ implies that $x < z$
$x < y$ implies that $x + z < y + z$ for any $z \in \mathbb{R}$
$x < y$ and $z > 0$ implies that $xz < yz$ (because $yz - xz = (y - x)z$ with $y - x, z \in \mathcal{P}$)
weak inequality: $a \leq b$ means $a < b$ or $a = b$; similar properties hold!

From these basic properties we can recover many familiar properties we have seen before; for example
$(-x)y = -(xy)$ $(-x)(-y) = xy$ $-(-x) = x$ $(x^{-1})^{-1} = x$
$x < 0$ and $y > 0$ implies $xy < 0$, $z < 0$ and $w < 0$ implies $zw > 0$
the additive inverse of a number is unique (i.e., if $x + y = 0 = x + z$ then $y = z$)

Proving things: Our ultimate goal is to provide proofs of some of the important results from calculus. This means that we need to justify the assertions we make, showing how a hypothesis forces our colnclusions to be true. Two often-used approaches:

Case analysis: Starting from a hypothesis (e.g., $x \neq 0$), one of several possibilities (cases) must be true (e.g., $x > 0$ or $x < 0$). If we show that in each case our hoped-for conclusion is true (e.g., $x^2 > 0$), then the hypothesis implies the conclusion ($x \neq 0$ implies $x^2 > 0$).

Proof by contradiction: "A implies B" is the same as "it is not possible for A to be true and also that B is false". Proof by contradiction consists of starting from 'A is true and B is false' and showing that we must inevitably show that something we know is false is true. This means that we cannot have A true and B false; so A implies B !

Example: using the Rational Roots Theorem (see below) we can show that it is not possible to have $x^3 = 5$ and $x \in \mathbb{Q}$. So $x^3 = 5$ implies $x \notin \mathbb{Q}$.

Another approach we will often use: induction! (see below)

Completeness: From the natural (= counting) numbers $\mathbb{N}$ we get the integers $\mathbb{Z}$ (by taking additive inverses) and then the rationals $\mathbb{Q}$ (by taking multiplicative inverses). But to get the reals $\mathbb{R}$ we need to step beyond the properties above.

A set $A \subseteq \mathbb{R}$ is *bounded (bdd) from above* if there is a $M \in \mathbb{R}$ so that $x \leq M$ for every $x \in A$.
A *least upper bound* λ is an upper bound for A so that no *smaller* number is an upper bound. In symbols: $x \leq \lambda$ for every $x \in A$ and if $\mu < \lambda$ then there is an $x \in A$ with $\mu < x$
[Equivalently: λ is an upper bd for A and if ν is *also* an upper bound for A then $\lambda \leq \nu$.]

Completeness Axiom: Every *non-empty* set $A \subseteq \mathbb{R}$ that is bdd from above *has* a least upper bound.

Least upper bound of A is unique! $\lambda = \sup(A)$

Application: If $x, y \in \mathbb{R}$ and $y - x > 1$, then there is an $n \in \mathbb{Z}$ with $x \leq n < y$.

Application: $A = \{x \in \mathbb{R} \ : \ x^2 < 2\}$ is non-empty and bdd above: $\lambda = \sup(A)$. Then we showed: $\lambda^2 = 2$ (!) So $\lambda =$ what we would call $\sqrt{2}$

Rational Roots Theorem: If $p(x) = a_0x^n + \cdots a_{n-1}x + a_n$ is a polynomial with integer coefficients $a_i \in \mathbb{Z}$ for all i), and if $r = \alpha/\beta$ is a rational root of p ($p(\alpha/\beta) = 0$ where α and β have no factors in common, *then* α evenly divides a_n and β evenly divides a_0.

Since $\sqrt{2}$ is a root of $p(x) = x^2 - 2$, which by the rational roots theorem has no rational roots, $\sqrt{2} \notin \mathbb{Q}$. (!) By the same reasoning, if $n \in \mathbb{N}$ and $\sqrt{n} \in \mathbb{Q}$ then $\sqrt{n} \in \mathbb{N}$.

Well-ordering Property: If $A \subseteq \mathbb{N}$ is non-empty, then it has a *smallest element.* That is, there is an $\lambda \in A$ so that *lambda* $\leq z$ for every $z \in A$. (In general for a set A whose infimum λ lies in A, we call it the *mminimum* of A. If the supremum lies in A, we call it the *maximum.*)

Application: the *Principle of Mathematical Induction:* If $A \subseteq \mathbb{N}$ is a set satisfying
(1) $n_0 \in A$, and (2) if $n \geq n_0$ and $n \in A$ then $n+1 \in A$, *then* $\{n \in \mathbb{N} \ : \ n \geq n_0\} \subseteq A$.

[Why? If not all $n \geq n_0$ are in A, then there is a smallest n which is not in A. But then $n - 1 \in A$ (or $n = n_0$) both of which contradict our hypotheses!]

PMI as it is usually stated: If $P(n)$ is a statement about the integer n so that
(1) $P(n_0)$ is true, and
(2) if $n \geq n_0$ and $P(n)$ is true then $P(n+1)$ must also be true,
then $P(n)$ is true for every integer $n \geq n_0$.

Sample applications:

For every integer $n \geq 1$ we have $\displaystyle\sum_{k=1}^{n} k \ = \ \frac{n(n+1)}{2}$

For every integer $n \geq 1$ we have $\displaystyle\sum_{k=1}^{n} k^2 \ = \ \frac{n(n+1)(2n+1)}{6}$

For every $n \geq 5$ we have $2^n > n^2$
Fibonacci sequence: $a_0 = a_1 = 1$ and for $n \geq 1$ we have $a_{n+1} = a_n + a_{n-1}$. Then for every $n \geq$ (something) we have $a_n \geq \left(\frac{3}{2}\right)^n$, and for every $n \geq 12$ we have $a_n \geq n^2$.

[For all of these the method by which we established the inductive step is probably more important as we go forward than the actual result!]

More Well-ordered property consequences:
If $\epsilon > 0$ then there is an $n \in \mathbb{N}$ so that $n\epsilon > 0$ (the Archimedean Principle).
[Add a positive number to itself enough times and you will get a big number!]
Rationals are everywhere: If $x, y \in \mathbb{R}$ with $x < y$ then there is an $r \in \mathbb{Q}$ so that $x < r < y$.

Sequences and convergence: Start with *distance*:
Absolute value: $|x| = x$ if $x \geq 0$, otherwise it is $-x$. So $|x| \geq 0$ for all x.
"Triangle inequality": $|x + y| \leq |x| + |y|$ for every $x, y \in \mathbb{R}$.
[Useful 'opposite' consquence: $|x - y| \geq |x| - |y|$
(useful for showing that $|x - y|$ is not small!]

$|x-y|$ = the distance between x and y. Triangle inequality: $|x-z| \leq |x-y| + |y-z|$

Sequences: A sequence is essentially just a function $f : \mathbb{N} \to \mathbb{R}$. We write $f(n) = a_n$

Examples: the Fibonnaci sequence! $a_0 = a_1 = 1$, $a_2 = 2$, $a_3 = 3$, $a_4 = 5$, $a_5 = 8$, $a_6 = 13$ ...

$a_n = 1 + (-1)^n/n$, $b_n = \dfrac{n^2 - n + 13}{3n^2 + 2n - 13}$

$a_n = \displaystyle\sum_{k=1}^{n} \frac{1}{k^2}$

Convergence: What happens to a sequence as n gets large? E.g., for $b_n = \dfrac{n^2 - n + 13}{3n^2 + 2n - 13}$, observation/intuition/calculus suggests that for large n b_n is approximately 1/3. Formalizing this notion leads us to:

$\lim_{n\to\infty} a_n = L$ means that a_n is close to L so long as n is large. More precisely, a_n is as close as we want it to be to L, so long as n is large enough. This leads us to

$\lim_{n\to\infty} a_n = L$ means for every $\epsilon > 0$ [think: small!] there is an $N \in \mathbb{N}$ [N depends on ϵ (!)] so that $n \geq N$ implies that $|a_n - L| < \epsilon$. If this limit exists (i.e., there is an L that the sequence converges to) we say that the sequence is *convergent*. If there is no number that the sequence converges to, we say that the sequence is *divergent*. [Shorthand: we write $a_n \to L$.]

The key here is that we wish to show that $|a_n - L|$ is small; we typically do this by comparing $|a_n - L|$ to other things that we know are small. This, in turn, we usually do by altering $|a_n - L|$, making it larger (but not too large!), until we end up with something we can show is small (so long as n is large enough!). To help us do this we have some basic limit results:

If $a_n \to L$ and $b_b \to M$, and $k \in \mathbb{R}$ is a konstant, then:
$ka_n \to kL$
$a_n + b_n \to L + M$
$a_n - b_n \to L - M$
$a_n b_n \to LM$
$a_n/b_n \to L/M$ (so long as $b_n, M \neq 0$)

Other properties we learned along the way:

If $(a_n)_{n=1}^{\infty}$ is convergent, then the set $A = \{a_n \ : \ n \in \mathbb{N}\}$ is bounded.
Limits are unique! If $a_n \to L$ and $a_n \to M$, then $L = M$.
$\dfrac{1}{n} \to 0$
If $a_n \geq a$ for every n (eventually!) and $a_n \to L$, then $L \geq a$
If $a_n \geq b_n$ for every n and $a_n \to L$, $b_n \to M$, then $L \geq M$
If $a_n \to L$ and $L \neq 0$, then eventually $|a_n| > \dfrac{|L|}{2}$

Showing convergence without knowing the limit: monotonicity

a_n is *monotone increasing* if $a_{n+1} \geq a_n$ for every n

b_n is *monotone decreasing* if $b_{n+1} \leq b_n$ for every n
If it is one or the other, we say the sequence is *monotone* (or *monotonic*).
A monotone increasing sequence that is bounded above (i.e., for some $M \in \mathbb{R}$ we have $a_n \leq M$ for every $n \in \mathbb{N}$) *converges* it limit is $\sup\{a_n \ : \ n \in \mathbb{N}\}$
[A monotone decreasing sequence that is bounded below also converges!]

Example:
$a_n = \sum_{k=1}^{n} \frac{1}{k^2} \leq 1 + \sum_{k=2}^{n} \frac{1}{k(k-1)} = 1 + \frac{n-1}{n} \leq 2$ (where the last equality is established by induction!), so $a_n \to L$ for some L [Fact: $L = \pi^2/6$ ...]

The Squeeze Theorem: If $a_n \leq b_n \leq c_n$ for every n and $a_n \to L$ and $c_n \to L$ as $n \to \infty$, then $b_n \to L$ and $n \to \infty$.

So, e.g., since $-1 \leq \sin n \leq 1$ for all n, $\frac{1}{n} \sin n \to 0$ as $n \to \infty$.

Subsequences: A subsequence amounts to choosing some of the terms of a sequence; formally, a subsequence is $a_{n_k} = a_{g(k)}$ for some strictly monotone increasing function $g : \mathbb{N} \to \mathbb{N}$. Subsequences inherit many of the same properties of the original sequence, for example, boundedness and convergence: if $a_n \to L$ then $a_{n_k} \to L$ for every subsequence of a_n. On the other hand,

Every bounded sequence has a monotonic subsequence. [But we cannot determine beforehand whether or not it will be increasing or descreasing.]

So a bounded sequence always has a convergent subsequence!