Math 325 Topics Sheet for Exam 2

[Technically, everything from the topics sheet for exam 1, plus...]

Subsequences: A subsequence amounts to choosing some of the terms of a sequence; formally, a subsequence is $a_{n_k} = a_{g(k)}$ for some strictly monotone increasing function $g : \mathbb{N} \to \mathbb{N}$. Subsequences inherit many of the same properties of the original sequence, for example, boundedness and convergence: if $a_n \to L$ then $a_{n_k} \to L$ for every subsequence of a_n. On the other hand,

Bolzano-Weierstrass Theorem: Every bounded sequence has a monotonic subsequence. [But we cannot determine beforehand whether or not it will be increasing or descreasing!]

Why? Try (e.g.) to find a monotonically *increasing* subsequence. If you succeed, done. If we always fail, then the sequence of last times that do work form a *decreasing* subsequence!

An 'intrinsic' characterization of convergence: **Cauchy sequences**
If $a_n \to L$, then the terms of the sequence eventually get close to one another: if $n \geq N$ implies $|a_n - L < \epsilon$ then $n, m \geq N$ implies

$$|a_n - a_m| = |(a_n - L) + (L - a_m)| \leq |a_n - L| + |L - a_m| = |a_n - L| + |a_m - L| < \epsilon + \epsilon = 2\epsilon$$

This leads to the notion of a *Cauchy sequence*:
$(a_n)_{n=1}^{\infty}$ is Cauchy if for every $\epsilon > 0$ there is an N so that $n, m \geq N$ implies $|a_n - a_m| < \epsilon$.

Convergent sequences are Cauchy.
More surprisingly, every Cauchy sequence is convergent! I.e., $(a_n)_{n=1}^{\infty}$ Cauchy implies that there is an L so that $a_n \to L$.

Finding L: Cauchy sequences are bounded. Then setting $b_n = \sup\{a_k \ : \ k \geq n\}$, the sequence b_n is monotone *decreasing*, and bounded below (by the lower bound of the entire sequence a_n), so b_n converges to some number L. Then $a_n \to L$, because for for any $\epsilon > 0$ there is an N so that for $n \geq N$ we have

$$|a_n - a_N| < \epsilon/3 \text{ and } |a_N - b_N| < \epsilon/3 \text{ and } |b_N - L| < \epsilon/3;$$

adding together gives $|a_n - L| < \epsilon$.

Functions of a real variable: Sequences are functions with domain $\mathbb{N}$. When we expand our allowed domains, $f : D \to \mathbb{R}$ for some $D \subseteq \mathbb{R}$, we can extend our notion of limit, as well.

$\lim_{x \to a} f(x) = L$ means that $|f(x) - L|$ is small, so long as $|x - a|$ is small enough. One feature: a need not be in the domain of f. If fact, even if is is, we do not care what value f takes there; our formal definition of the limit is

For every $\epsilon > 0$, there is a $\delta > 0$ so that $x \in D$ and $0 < |x - a| < \delta$ implies that $|f(x) - L| < \epsilon$.

One sticky point: such limits need not be unique! [If $x \in D$ and $0 < |x - a| < \delta$ is satified by no number, then L could be anything we want!] For that matter, the limit need not exist! If it does, we say that f converges at a. For this purpose, we generally restrict ourselves, in discussing limits, to *accumulation points* of D. c is an accumulation point of D if for every $\delta > 0$ there is an $x \in D$ with $0 < |x - c| < \delta$. [That is, no matter how close to c we need to be, there are points of D other than c that are at least that close.] With this, if c is an accumulation point of D, then the limit of f at c (if it exists) is unique. One special case: $(c-\delta, c+\delta) \subseteq D$ for some $\delta > 0$.

Leveraging our work on sequences, $\lim_{x\to a} f(x)$ can be computed using sequences. $\lim_{x\to a} f(x) = L$ if and only if for every sequence a_n with $a_n \to a$ and $a_n \neq a$ for every n, we have $f(a_n) \to L$.

Then most of our familiar results about limits of sequences carry over to functions: for example if $f(x) \to L$ and $g(x) \to M$ as $x \to a$, then $(f+g)(x) \to L+M$ and $(f \cdot g)(x) \to LM$.

Continuity: From calculus you are used to the idea that for many functions to compute its limit we "plug in". That is, $\lim_{x\to c} f(x) = f(c)$. We call such a function *continuous at c*. If it is not continuous at c, we say it is *discontinuous* at c. If $f : D \to \mathbb{R}$ is continuous at c for every $c \in D$, we say that it is continuous on D. Continuity can be described using ϵ's and δ's: f is continuous at c provided that for every $\epsilon > 0$ there is a $\delta > 0$ so that $|x - c| < \delta$ implies that $|f(x) - f(c)| < \epsilon$.

Because limits can be described using sequences, so can continuity. In particular we have that f is continuous at $c \in D$ if and only if for every sequence $a_n \to c$ with $a_n \in D$ for every n, we have $f(a_n) \to f(c)$.

This enables us to use results about sequences to prove results about continuous functions. For example, if f and g are both continuous at c then so are $f+g$, $f-g$, $f \cdot g$, and f/g (so long, for the last, as $g(c) \neq 0$).

Possibly the two most important results about continuous functions are:

Intermediate Value Theorem: If $f : [a,b] \to \mathbb{R}$ is continuous on $[a,b]$ and D lies between $f(a)$ and $f(b)$, then there is a $c \in [a,b]$ so that $f(c) = D$.

Extreme Value Theorem: If $f : [a,b] \to \mathbb{R}$ is continuous on $[a,b]$, then there are $c, d \in [a,b]$ so that $f(c) \leq f(x) \leq f(d)$ for every $x \in [a,b]$.

The IVT can be used in root-finding: if f is continuous on an interval and $f(\alpha) < 0 < f(\beta)$, then there is a root of f lying between α and β. By repeatedly narrowing the distance between α and β (like, for example, taking their midpoint), we can find succesively better approximations to the root.

The IVT also allows us to show that every (positive) real number has an n-th root, for any natural number n; $f(x) = x^n - c$ always has a root.

The EVT tells us that maxima and minima exist, for function defined on a closed interval. [Techniques of calculus tells us how to find them, for differentiable functions.]

Inverse functions. Functions that are one-to-one have inverses. A continuous function $f : I \to \mathbb{R}$ that is one-to-one must be either monotonically increasing or monotonically decreasing (in the 'strong' sense: we cannot have $x < y$ and $f(x) = f(y)$). [This has a rather tedious proof...]. But more importantly, as a result a one-to-one continuous function has a continuous inverse. This is because if (say) f is increasing, then $g = f^{-1}$ is also increasing, and given $a \in f(I)$ and $\epsilon > 0$, we have $f(g(a) - \epsilon) < f(g(a)) = a < f(g(a) + \epsilon)$, so there is a $\delta > 0$ with $f(g(a) - \epsilon) < a - \delta < a + \delta < f(g(a) + \epsilon)$, so $|x - a| < \delta$ means

$a - \delta < x < a + \delta$, so $f(g(a) - \epsilon) < f(g(x)) < f(g(a) + \epsilon)$, so (since g is increasing!) $g(a) - \epsilon = g(f(g(a) - \epsilon)) < g(f(g(x))) = g(x) < g(f(g(a) + \epsilon)) = g(a) + \epsilon$, so $g(a) - \epsilon < g(x) < g(a) + \epsilon$, that is, $|g(x) - g(a)| < \epsilon$.

This, in turn, tells us that many of our favorite functions are continuous. Since $f(x) = x^n$ is continuous, it is one-to-one (for $x \geq 0$ if n is even), its inverse $g(x) = x^{1/n}$ is continuous. Also, for example, $f(x) = x^5 + 5x^3 + 17x - 4$ is continuous and one-to-one (by calculus, or directly comparing output for $x < y$), so it's inverse, which we (probably) can't express in an 'elementary' way, is continuous!

Uniform Continuity: In many situations, continuity alone is not 'enough' to obtain the results that we might want. For example, for each $x \in [0, 1]$ $f_n(x) = x^n \to 0$ if $x < 1$ and $\to 1$ if $x = 1$. Each of the functions involved is continuous, but their 'limit' is not! The 'problem' is that continuity is defined for each point: the $\delta > 0$ we find is chosen with knowledge of both $\epsilon > 0$ and the point $c \in D$ at which continuity is being studied. So δ is a function of both ϵ and c.

A stronger form of continuity is obtained by eliminating one of these dependences: $f : D \to \mathbb{R}$ is *uniformly continuous* on D if for every $\epsilon > 0$ there is a $\delta > 0$ so that $x, y \in D$ and $|x - y| < \delta$ implies that $|f(x) - f(y)| < \epsilon$. That is, δ depends ony on ϵ, not on which points are input to f. A uniformly continuous function is therefore continuous, but the opposite need not be true. $f(x) = 1/x$ is continuous on the interval $(0, \infty)$, but is not uniformly continuous on that interval.

But if the domain of f is a closed interval $[a, b]$, then continuity does imply uniform continuity. [Our proof relied on the fact that bounded sequences have convergent subsequences!]

Uniform continuity is an important component of many of the results we will study for the remainder of the semester!