Math 325 Topics Sheet for the "last" exam

[Technically, everything from the topics sheets for the first two exams, <u>plus</u>...]

Differentiation: Now that we have put limits of functions on a firm foundation, we can take a more precise look at *differential calculus.*

A fcn $f : I \to \mathbb{R}$ is *differentiable* at $a \in I$ if $\lim\limits_{x\to a} \dfrac{f(x)-f(a)}{x-a}$ exists; we call the limit $f'(a)$. This means:

Given any $\epsilon > 0$, we can find $\delta > 0$ so that $0 < |x-a| < \delta$ and $x \in I$ implies $|\dfrac{f(x)-f(a)}{x-a} - f'(a)| < \epsilon$.

If f is differntiable at evey $a \in I$ we say that f is differentiable on I. Using the familiar variation $f'(a) = \lim\limits_{h\to 0} \dfrac{f(a+h)-f(a)}{h}$, we can treat $f'(a)$ as a function of a and write the derivative function $f'(x) = \lim\limits_{h\to 0} \dfrac{f(x+h)-f(x)}{h}$, whose (implied) domain is everywhere that the limit exists.

The properties of limits then give us:

If f is differentiable at a, then f is continuous at a ($f(x) - f(a) = \dfrac{f(x)-f(a)}{x-a}(x-a)$).

$(f+g)'(x) = f'(x) + g'(x)$ (where both sides are defined)

$(cf)'(x) = cf'(x)$ (when one side is defined, then the other one is)

$(fg)'(x) = f'(x)g(x) + f(x)g'(x)$ (where both sides are defined)

$(f/g)\prime(x) = (f'(x)g(x) - f(x)g'(x))/(g(x))^2$ (where both sides are defined)

These follow the same line of argument your calculus instructor would have followed.

But one differentiation rule that requires more care is the *Chain Rule*: if g is differentiable nt $x = a$ and f is differentiable at $g(a)$, then $f \circ g$ is differentiable at $x = a$ and $(f \circ g)'(a) = f'(g(a)) \cdot g'(a)$.

This is because the standard line of argument, writing $\dfrac{(f\circ g)(x) - (f\circ g)(a)}{x-a} = \dfrac{(f\circ g)(x) - (f\circ g)(a)}{g(x)-g(a)}$ g(

<u>assumes</u> that $g(x) \neq g(a)$ for x near a. This is <u>true</u> if $g'(a) \neq 0$:

If $g'(a) \neq 0$, then $\dfrac{g(x)-g(a)}{x-a} \neq 0$ near a (and in fact has the same sign as $g'(a)$, near a).

But! if $g'(a) = 0$, we need to appeal to the $\epsilon - \delta$ definition directly, to show that $(f \circ g)'(a) = 0$.

The 'other' inverse function theorem: if f is one-to-one <u>and</u> differentiable on an interval I, then the inverse function $g(x) = f^{-1}(x)$ is also differentiable (on $f(I)$), and $g'(x) = 1/f'(g(x))$.

This follows directly from the chain rule (since $f(g(x)) = x$), <u>if</u> we know that g is differentiable! But this, again, requires a direct appeal to the $\epsilon - delta$ definition, by showing that

$$\frac{g(f(a)+h) - g(f(a))}{h} = \frac{k}{f(a+k) - f(a)}, \text{ where } k = k(h) = g(f(a)+h) - a \ (!)$$

Then since $h \to 0$ implies that $k \to 0$ (since g is continuous!), we get our result.

The main result that makes differentiation so useful, and allows the derivative of f to tell us useful things about f, is the

Mean Value Theorem: If $f : [a, b] \to \mathbb{R}$ is continuous on $[a, b]$ and differentiable on (a, b)¡ then there is a(t least one) $c \in (a, b)$ with $f'(c) = \dfrac{f(b) - f(a)}{b - a}$.

The proof of MVT uses that fact that (1) at a max or c of a differentiable function (except at an endpoint of the domain), $f'(c) = 0$, (2) if $f(a) = f(b)$ then one of the absolute max or absolute min c of f (which exist by EVT) lies in (a, b) (so $f'(c) = 0$: this is *Rolle's Theorem*), and (3) applying Rolloe's Thoerem to the function

$$g(x) = f(x) - (f(a) + \frac{f(b) - f(a)}{b - a}(x - a)$$

yields a $c \in (a, b)$ with $g'(c) = 0 = f'(c) - \frac{f(b)-f(a)}{b-a}$.

This is the result which tells us most of the important things we know about the derivative:

If $f'(x) > 0$ on an interval I, then f is increasing on I (recall that we shoed that $f'(a) > 0$ does not imply that f is increasing near a...)
If $f'(x) < 0$ on an interval I, then f is decreasing on I
If $f'(x) = 0$ on an interval I, then f is constant on I
If $f'(x) = g'(x)$ on an interval I, then for some constant $c \in \mathbb{R}$ we have $g(x) = f(x) + c$ on I.

A more sophisticated version of MVT will allow us to prove L'Hôpital's Rule. The *Cauchy Mean Value Theorem* asserts that if $f, : [a, b] \to \mathbb{R}$ are continuous functions on $[a, b]$, and are both differentiable on (a, b), then there is a $c \in (a, b)$ so that $(f(b) - f(a))g'(c) = (g(b) - g(a))f'(c)$. It follows from Rolle's Theorem, applied to the function $h(x) = (f(b) - f(a))g(x) - (g(b) - g(a))f(x)$

With it, we can prove the 'real' L'Hôpital's Rule: If $f(x), g(x) \to 0$ as $x \to a$, f and g are differentiable near a, and $\dfrac{f'(x)}{g'(x)} \to L$ as $x \to a$, then $\dfrac{f(x)}{g(x)} \to L$ as $x \to a$.

This allows us to 'iteratively' apply L'Hôpital, to compute limits like $\displaystyle\lim_{x\to 0} \frac{f(x)}{g(x)} = \lim_{x\to 0} \frac{\sin x - (x - \frac{x^3}{6} + \frac{x^5}{12}$ as $\displaystyle\lim_{x\to 0} \frac{f^{(6)}(x)}{g^{(6)}(x)}$ [working backwards to invoke L'Hôpital to show all of the intervening limits exxist and a equal!].

Integration: One use of uniform continuity arises in the construction of the integral of a function. If $f : [a, b] \to \mathbb{R}$ is bounded, then we can try to approximate the area under the graph of f using rectangles. If the rectangles all lie inside this region, then our intuition tells us that the sum of their areas will be less than the area of the region. If the rectangles completely cover the region, then the sum of their areas will be greater than the area of the region. These ideas underlie the formal notion of the *Riemann integral*.

But our first approach allows us to pick the 'heights' of our rectangles arbitrarily, using function values. Here the idea is that if all of the rectangles are 'thin' enough, and the heights of the rectangles 'approximate' f, then the sum of their areas should 'approximate' the area under the graph of f. Formally, we say that a function $f : [a, b] \to \mathbb{R}$ is integrable on $[a, b]$ if there is a number L so that for any choice of *partition* $P = \{a = x_0 < x_1 < \cdots < x_n = b$, so long as the widths $x_i - x_{i-1}$ of the subintervals are all small, then any choice of $c_i \in [x_{i-1}, x_i]$ (a 'sampling', S) has $\sum_{i=1}^{n} f(c_i)(x_i - x_{i-1})$ close to L. Setting $||P||$ =max$\{x_i - x_{i-1} \ : \ i = 1, \ldots, n\}$ = the norm of the partition P, wht we want is that for any $\epsilon > 0$ there is a $\delta > 0$ so that for any partition P with $||P|| < \delta$ and for any sampling S for P, we have $|R(f, P, S) - L| = |\sum_{i=1}^{n} f(c_i)(x_i - x_{i-1}) - L| < \epsilon$. We call $R(f, P, S)$ a *Riemman sum* for f on $[a, b]$ with partition P and sampling S.

If we can show this for some L, then we define $\int_a^b f(x)\, dx = L$. For example, we can show directly from this definition that $\int_0^2 x = 2$, by showing, for $f(x) = x$ and any partition P of $[0, 2]$, that with $S = \{\frac{x_{i-1}+x_i}{2}\} = \{c_i\}$, we have $R(f, P, S) = 2$, and for any other sampling $S' = \{d_i\}$ of P, if $||P|| < \delta$ then $|R(f, P, S') - 2| = |R(f, P, S') - R(f, P, S))| \leq \sum |f(c_i) - f(d_i))| \cdot |x_i - x_{i-1}| = \sum |c_i - d_i| cdot |x_i - x_{i-1}| < \sum \delta |x_i - x_{i-1}| = 2\delta < \epsilon$¡ if we choose $\delta = \epsilon/2$.

But, in practice, we often wish to establish the f is integrable without knowing what L is. This we can do, since if $R(f, P, S)$ and $R(f, Q, T)$ are both close to L, then they are close to one another. This leads to a 'Cauchy-like' condition:

f is integrable on $[a, b]$ if and only if, (*) for every $\epsilon > 0$, we can find a $\delta > 0$ so that for any two partitions P, Q with $||p||, ||q|| < \delta$, and for any pair of samplings, S for P and T for Q, we have $|R(f, P, S) - R(f, Q, T)| < \epsilon$. [The integral L can then be found by taking any collection of partitions P_n with $||P_n|| \to 0$ and any samplings S_n (e.g., left endpoint, or right endpoints, or midpoints! of the intervals) for P_n; $L = \lim_{n\to\infty} R(f, P_n, S_n)$.

But! dealing with two partitions at once poses certain technical challenges, which get in the way of our effectively using this condition. What we would like to do insteead is to show that integrability can be established by looking at one partition at a time; we would like to show that it is implied by:

(**) for any $\epsilon > 0$, we can find a $\delta > 0$ so that for any partition P with $||P|| < \delta$, and any pair of samplings S, T for P, we have $R(f, P, S) - R(f, P, T)| < \epsilon$.

This turns out to work! To see this, we need to introduce the upper Riemann sum $U(f, P)$ and lower Riemnn sum $L(f, P)$: given a partition P of the interval $[a, b]$) we can define $m_i = \inf\{f(x) \ : \ x \in [x_i, x_{i+1}]\}$ and $M_i = \sup\{f(x) \ : \ x \in [x_i, x_{i+1}]\}$. Then $m_i \leq f(c_i) \leq M_i$ for any choice of sampling, so $L(f, P) = \sum m_i(x_i - x_{i-1}) \leq \sum f(c_i)(x_i - x_{i-1}) = R(f, P, S) \leq \sum M)i(x_i - x_{i-1}) = U(f, P)$

Since we can always choose $\{c_i\} = S$ so that $M_i - f(c_i)$ is as small as we like (by choosing values close to the sup), and we can choose $\{d_i\} = T$ so that $f(d_i) - m_i$ is as small as we like, as can make the finite sum $\sum(f(c_i) - f(d_i)(x_i - x_{i-1}) = R(f, P, S) - R(f, P, T)$ as close to $\sum(M_)i - m_i)(x_i - x_{i-1}) = U(f, P) - L(f, P)$ as we like, so we can show that that (**) holds $\Leftrightarrow$ for every $\epsilon > 0$ there is a $\delta > 0$ so that for any partition P with $||P|| < \delta$ we have $U(f, P) - L(f, P) < \epsilon$.

But for partitions $P \subseteq Q$ we have $L(f,P) \leq L(f,Q) \leq U(f,Q) \leq U(f,P)$, since points of Q subdivide some of the suintervals for P, and sup's over smaller intervals go down (so $U(f,Q) \leq U(f,P)$) while inf's for subintervals go up (so $L(f,P) \leq L(f,Q)$).

With this in hand, we can show that (**) implies our 'Cauchy' condition (*): given two partitions P, Q and samplings S, T, $R(f,P,S) - R(f,Q,T) \leq U(f,P) - L(f,Q) = (U(f,P) - L(f,P)) + L(f,P) - U(f,Q) + (U(f,Q) - L(f,Q))$. But then since $U(f, P \cup Q) - L(f, P \cup Q) \geq 0$, we have $R(f,P,S) - R(f,Q,T) \leq (U(f,P) - L(f,P)) + (L(f,P) - L(f, P \cup Q)) + (U(f, P \cup Q) - U(f,Q)) + (U(f,Q) - L(f,Q))$. But! $L(f,P) - L(f, P \cup Q) \leq 0$ and $U(f, P \cup Q) - U(f,Q) \leq 0$, since $P, Q \subseteq P \cup Q$. So $R(f,P,S) - R(f,Q,T) \leq (U(f,P) - L(f,P)) + (U(f,Q) - L(f,Q))$ If we pick a $\delta > 0$ so that $||P|| < \delta$ implies that $U(f,P) - L(f,P) < \epsilon/2$, then we have:

$||P||, ||Q|| < \delta$ implies that $R(f,P,S) - R(f,Q,T) < \epsilon/2 + \epsilon/2 = \epsilon$. Reversing roles of P and Q (and running the same argument again) yields $R(f,Q,T) - R(f,P,S) < \epsilon$, so $||P||, ||Q|| < \delta$ implies that $|R(f,P,S) - R(f,Q,T)| < \epsilon$, so (**) implies the 'Cauchy' integrability criterion, so f is integrable!

Unfortunately, not every function is Riemann integrable! For example, our favorite terrible function, $f(x) = 0$ if $x \in \mathbb{Q}$ and $f(x) = 1$ if $x \notin \mathbb{Q}$, is integrable over no interval (containing more than one point). <u>But</u>:

If f is continuous on $[a,b]$, then f is integrable on $[a,b]$. We can show, using that fact that f is also <u>uniformly</u> continuous, (using the δ from uniform continuity associated to $\epsilon/(b-a) > 0$ that $||P|| < \delta$ implies $M_i - m_i < \epsilon$ for all i, so $U(f,P) - L(f,P) < \sum[\epsilon/(b-a)] \cdot (x_i - x_{i-1}) = [\epsilon/(b-a)] \cdot (b-a) = \epsilon$

From these different ways of approaching integrability, we can establish some familiar integration results:

If f is integrable on $[a,b]$ and $a < c < b$, then f *is integrable on both* $[a,c]$ *and* $[c,b]$ *and* $\int_a^b f(x)\,dx = \int_a^c f(x)\,dx \int_a^b f(x)\,dx$. [The italicised part needed Riemann sums to establish rigorously; then we could choose any partition we wanted to compute the integrals.]

If f and g are integrable on $[a,b]$, then *both* $c \cdot f$ *and* $f + g$ *are integrable on* $[a,b]$, *and* $\int_a^b (cf)(x)\,dx = c\int_a^b f(x)\,dx$ and $\int_a^b (f+g)(x)\,dx = \int_a^b f(x)\,dx + \int_a^b g(x)\,dx$.

Most times we do not compute integrals using Riemann sums! Instead we rely on

The (First) Fundamental Theorem of Calculus: If $f : [a,b] \to \mathbb{R}$ is integrable, and F is an antiderivative of f, so $F'(x) = f(x)$ on the interval, then $\displaystyle\int_a^b f(x)\,dx = F(b) - F(a)$

Basic question: which functions <u>have</u> antiderivatives?

$f'(x) = \displaystyle\lim_{h \to 0} \frac{f(x+h) - f(x)}{h} = \lim_{z \to x} \frac{f(z) - f(x)}{z - x}$. Domain = those x for which the limit <u>exists</u>. Limit exist at a: we say f is *differentiable* at a. f is *differentiable on* D if it is differentiable at every point of D.

If f is differentiable at a then f is continuous at a.

Intermediate Value Theorem for Derivatives: If f is differentiable on $[a, b]$, then for every $c, d \in [a, b]$ if γ is between $f'(c)$ and $f'(d)$ then there is a w between c and d so that $f'(w) = \gamma$.

This result was established by first proving an analogous result about slopes of secants: if f is continuous on an interval I, then the set $S = \{\dfrac{f(x) - f(y)}{x - y} \ : \ x < Y \text{ and } x, y \in I\}$ is an interval in $\mathbb{R}$. The point is that f' need not be continuous for this to be true! In fact, though, this result tells us a lot about how f' can fail to be continuous:

If g is not cts at a, then either $\lim\limits_{x\to a}$ exists but is not equal to $f(a)$ [hole], or $\lim\limits_{x\to a^-}$ and $\lim\limits_{x\to a^+}$ both exist but are not equal [jump], or one of these one-sided limits fails to exist [oscillation]. The result above implies that if f is differentiable on $[a, b]$ but f' fails to be cts at some c, then this failure must be by oscillation. So where f' fails to be cts, it really fails...

The other point is that a function (like one with a jump discontinuity) that fails the IVT4Derivs cannot have an antiderivative! So the Fund Thm cannot be applied to compute its integral...

But lots of our favorite functions do have antiderivatives:

The (Second) Fundamental Theorem of Calculus: If f is continuous on $[a, b]$, then the function $F(x) = \int_a^x f(t)\ dt$, defined on $[a, b]$, has $F'(x) = f(x)$ on $[a, b]$.

Being able to 'manufacture' antiderivatives in this way allows us to create new functions!, which turn out to be very useful to us in a variety of ways.

For example, the 'right' way to introduce exponential and logarithmic functions is to start with a familiar function, like $f(x) = 1/x$, and then (using that we 'know' that $(\ln x)' = 1/x$ and $ln(1) = 0$) construct the logarithm as $\ln x = \displaystyle\int_1^x \frac{1}{t}\ dt$. This has domain $(0, \infty)$ (the larget interval on which $f(x) = 1/x$ is continuous) and has derivative $(\ln x)' = 1/x > 0$ on that interval, so $\ln x$ is increasing, so it has a (continuous and differentiable) inverse function, which we call $exp(x) = e^x$ (!). All of the familiar properties of logarithms can be deduced from this definition, using the properties of integrals (and derivatives).