Math 325 Exam 1 practice problems

1. (25 pts.) Show that if $x, y \geq 0$, then the *arithmetic mean* $m = \dfrac{x+y}{2}$ and the *geometric mean* $\mu = \sqrt{xy}$ of x and y always satisfies $m \geq \mu$.

   [Hint: show that $2(m-\mu)$ is a square!]

   Show by an example that this inequality can be <u>strict</u> (that is, $m > \mu$).

2. (20 pts.) Show that $\alpha = \sqrt{2+\sqrt{7}}$ is **not** a rational number.

3. (30 pts.) We will define a sequence $(a_n)_{n=1}^{\infty}$ by setting $a_1 = 2$, and for $n \geq 1$ (inductively) setting

   $$a_{n+1} = 3 + \sqrt{2a_n}\ .$$

   Show that this sequence is both monotonically increasing and bounded from above (so the sequence converges).

4. (25 pts.) Given sequences $(a_n)_{n=1}^{\infty}$ and $(b_n)_{n=1}^{\infty}$, show that <u>if</u> the sequences

   $$c_n = a_n + b_n \qquad \text{and} \qquad d_n = a_n - b_n$$

   both converge, <u>then</u> the sequences a_n and b_n <u>also</u> both converge!

5. Use induction to show that for every $n \geq 1$,

   $$a_n = \sum_{k=1}^{n} \frac{1}{k(k+2)} = \frac{n(3n+5)}{4(n+1)(n+2)} = f(n)$$

   (Hint: write out what $f(n+1)$ is; it will help.)

6. Use the rational roots theorrem to show that $\alpha = \sqrt{2} - \sqrt{5}$ is not a rational number.

7. Find the limit of the sequence $a_n = \dfrac{n^2-n+1}{3n^2-1}$

   and prove you are right using the $\epsilon - N$ definition of the limit. [Also: how how to do this quicker using our limit theorems!]