Math 325, Section 1

Exam 1 Practice Exam Solutions

1. Show that if $x, y \geq 0$ the n the *arithmetic mean* $m = \dfrac{x+y}{2}$ and the *geometric mean* $\mu = \sqrt{xy}$ always satisfies $m \geq \mu$. Show by an example that this inequality can be strict.

   If we compute $2(m-\mu) = 2(\dfrac{x+y}{2} - \sqrt{xy}) = x - 2\sqrt{xy} + y = (\sqrt{x}^2 - 2\sqrt{x}\sqrt{y} + (\sqrt{y})^2 = (\sqrt{x} - \sqrt{y})^2$, then we find, in particular, that $2(m-\mu) \geq 0$, so $m - \mu \geq 0$, so $m \geq \mu$, as desired.

   Essentially any pair of (distinct!) non-negative number will have $m > \mu$; for example, $x = 1$ and $y = 9$ give $m = 5$ and $\mu = 3$, and $5 > 3$.

2. Show, using the Rational Roots Theorem, that $\alpha = \sqrt{2 + \sqrt{7}}$ is **not** a rational number.

There are (at least) two ways to show this. Via the Rat'l Roots Thm, we find a polynomial having α as a root:

$\alpha^2 = 2 + \sqrt{7}$, so $\alpha^2 - 2 = \sqrt{7}$, so $(\alpha^2 - 2)^2 = 7$, so

$$(\alpha^2 - 2)^2 - 7 = \alpha^4 - 4\alpha^2 + 4 - 7 = \alpha^4 - 4\alpha^2 - 3 = 0.$$

So α is a root of the polynomial $p(x) = x^4 - 4x^2 - 3$. But the Rat'l Roots Thm. tells us that the only possible rational roots of this polynomial are $1, -1, 3$, and/or -3. But we can either plug all of these into p and note that none of them are roots of p (this is probably the preferred way?), or we can be a little sneakier. Note that $\alpha^2 = 2 + \sqrt{7} > 2 + \sqrt{4} = 2 + 2 = 4$, so $\alpha > 2$, but $\alpha^2 = 2 + \sqrt{7} \leq 2 + \sqrt{9} = 2 + 3 = 5 < 9$, so $\alpha < 3$. So α cannot be equal to any of these possible roots. In either case we then know that α, which is a root of p, cannot be equal to any of the possible rational roots of p, so α cannot be rational!

Alternate proof: suppose $\alpha = p/q$ is rational. Then $\alpha^2 = p^2/q^2$ is also rational, so $\alpha^2 - 2 = (p^2 - 2q^2)/q^2$ is rational. But! by the work above, $\alpha^2 - 2 = \sqrt{7} = \beta$ is then rational. But β is a root of $r(x) = x^2 - 7$, whose only possible rational roots, $1, -1, 7, -7$, aren't roots! So β isn't rational. But if α is rational so is β ! So α cannot be rational.

3. We will define a sequence $(a_n)_{n=1}^{\infty}$ by setting $a_1 = 2$, and for $n \geq 1$ (inductively) setting

$$a_{n+1} = 3 + \sqrt{2a_n} \ .$$

   Show that this sequence is both monotonically increasing and bounded from above (so the sequence converges).

$a_2 = 3 + \sqrt{2 \cdot 2} = 3 + \sqrt{4} = 3 + 2 = 5 \geq 2 = a_1$, so $a_2 \geq a_1$, which gets us started on an induction. If we now suppose (as our inductive hypothesis) that $a_{n+1} \geq a_n$, then $2a_{n+1} \geq 2a_n$ (since $2a_{n+1} - 2a_n = 2(a_{n+1} - a_n)$ is the product of a positive number (2) and a non-negative one). But then $\sqrt{2a_{n+1}} \geq \sqrt{2a_n}$, from a result in class, and so $a_{n+2} = 3 + \sqrt{2a_{n+1}} \geq 3 + \sqrt{2a_n} = a_{n+1}$.

So $a_{n+1} \geq a_n$ implies that $a_{n+2} \geq a_{n+1}$, giving our inductive step. So $a_{n+1} \geq a_n$ for every $n \geq 1$, by induction.

To show that the sequence is bounded, we could just pick an impossibly large number and give it a try. Or we could use techniques like we have before to find out when $M = 3+\sqrt{2M}$, and use that. Or we could note that the thing which controls the size of a_{n+1} is $\sqrt{2a_n}$, which for a_n "large" is a lot smaller than a_n, for example, $a_n = 50$ gives $a_{n+1} = 3 + \sqrt{100} = 13$, which is a lot smaller than 50.

So let's pick $M = 50$, say, and show that $a_n \leq 50$ for every n, by induction! $a_1 = 2 \leq 50$ is true, so our base case works. Then if $a_n \leq 50$, then $2a - n \leq 100$¡ so $\sqrt{2a_n} \leq \sqrt{100} = 10$, so $a_{n+1} = 3 + \sqrt{2a_n} \leq 3 + 10 = 13 \leq 50$. This is our inductive step; $a_n \leq 50$ implies that $a_{n+1} \leq 50$. So $a_n \leq 50$ for all $n \geq 1$, by induction; so the sequence is bounded above.

Because it is a monotone increasing sequence which is bounded above, it then follows that the sequence converges.

[N.B.: We can, in fact, find the limit of the sequence; as with examples from class our limit properties allow us to conclude that the limit, L, satisfies $L = 3 + \sqrt{2L}$, so $(L-3)^2 - 2L = L^2 - 8L + 9 = 0$. Using the quadratic formula, we conclude that

$$L = (8 \pm \sqrt{64-36})/2 = (8 \pm 2\sqrt{7})/2 = 4 \pm \sqrt{7}.$$

Since $L \geq a_2 = 5$ (since $a_n \geq a_2$ for every $n \geq 2$) and $4 - \sqrt{7} \leq 4 - \sqrt{4} = 4 - 2 = 2$, we conclude that $L = 4 + \sqrt{7}$.]

4. Given sequences $(a_n)_{n=1}^{\infty}$ and $(b_n)_{n=1}^{\infty}$, show that <u>if</u> the sequences

$$c_n = a_n + b_n \qquad \text{and} \qquad d_n = a_n - b_n$$

both converge, <u>then</u> the sequences a_n and b_n <u>also</u> both converge!

Since c_n and d_n both converge, we know that $c_n + d_n = (a_n + b_n) + (a_n - b_n) = 2a_n$ also converges. So $a_n = (1/2)(2a_n)$ also converges!

But then a_n and $c_n = a_n + b_n$ converge, and so $c_n - a_n = (a_n + b_n) - a_n = b_n$ must converge, as well. So both $(a_n)_{n=1}^{\infty}$ and $(b_n)_{n=1}^{\infty}$must be convergent sequences.

A somewhat different way to write the same thing is:

If $c_n = a_n + b_n \to L$ and $d_n = a_n - b_n \to M$, then $c_n + d_n = 2a_n \to L + M$, so $a_n = (1/2)(2a_n) \to (1/2)(L + M)$. In particular a_n has a limit, so it converges! Then $b_n = (a_n + b_n) - a_n \to L - (1/2)(L + M) = (1/2)(L - M)$, so b_n has a limit, so b_n converges!

[There are several other, roughly equivalent, ways to see how to build a_n and b_n out of c_n and d_n, leading to the same conclusions.]

5. Use induction to show that for every n≥1,

$$a_n = \sum_{k=1}^{n} \frac{1}{k(k+2)} = \frac{n(3n+5)}{4(n+1)(n+2)} = \text{f}(n).$$

(Hint: write out what f(n+1) is; it'll help.

Our base case is $n = 1$, and we have $a_1 = \dfrac{1}{(1)(1+2)} = \dfrac{1}{3} = \dfrac{8}{4\cdot 2\cdot 3} = \dfrac{(1)(3\cdot 1+5)}{4(1+1)(1+2)}$, as desired.

For the inductive step, if we suppose that $a_n = \displaystyle\sum_{k=1}^{n} \frac{1}{k(k+2)} = \frac{n(3n+5)}{4(n+1)(n+2)}$, then $a_{n+1} = a_n + \dfrac{1}{(n+1)(n+3)} = \dfrac{n(3n+5)}{4(n+1)(n+2)} + \dfrac{1}{(n+1)(n+3)}$. Putting over a common denominator, this last sum is equal to

$$\frac{n(3n+5)(n+3)}{4(n+1)(n+2)(n+3)} + \frac{4(n+2)}{4(n+1)(n+2)(n+3)} = \frac{n(3n^2+5n+9n+15)+4n+8}{4(n+1)(n+2)(n+3)} = \frac{3n^3+14n^2+19n+8}{4(n+1)(n+2)(n+3)} = \frac{(n+1)(3n^2+11n+8)}{4(n+1)(n+2)(n+3)} = \frac{3n^2+11n+8}{4(n+2)(n+3)} = \frac{(3n+8)(n+1)}{4(n+2)(n+3)} = \frac{(n+1)(3(n+1)+5)}{4((n+1)+1)((n+1)+2)} = f(n+1)\ .$$

So we have shown that $a_1 = f(1)$, and $a_n = f(n)$ implies that $a_{n+1} = f(n+1)$. So, by induction, we have shown that $a_n = f(n)$ for every $n \in \mathbb{N}$ with $n \geq 1$, as desired.

An alternate approach: Noting that $\dfrac{1}{k(k+2)} = \dfrac{1}{2}(\dfrac{1}{k} - \dfrac{1}{k+2})$, we can show (by induction!) that $a_n = \frac{1}{2}(\frac{1}{1} + \frac{1}{2} - \frac{1}{(n+1)} - \frac{1}{(n+2)})$ (*), since (check!) this is true for $n = 1$, and then in the inductive step

$a_{n+1} = \frac{1}{2}(\frac{1}{1}+\frac{1}{2}-\frac{1}{(n+1)}-\frac{1}{(n+2)}+\frac{1}{(n+1)(n+3)} = \frac{1}{2}([\frac{1}{1}+\frac{1}{2}-\frac{1}{(n+1)}-\frac{1}{(n+2)}]+\frac{1}{(n+1)}-\frac{1}{(n+3)}) = \frac{1}{2}(\frac{1}{1}+\frac{1}{2}--\frac{1}{(n+2)}-\frac{1}{(n+3)})$, as desired.

Putting the expression (*) over a common denominator yields the result.

6. Use the Rational Roots Theorem to show that $r = \sqrt{2} - \sqrt{5}$ is a not a rational number.

If $r = \sqrt{2} - \sqrt{5}$, then $\alpha^2 = (\sqrt{2}-\sqrt{5})^2 = 2 - 2\sqrt{2}\sqrt{5} + 5 = 7 - 2\sqrt{10}$, Then $r^2 - 7 = 2\sqrt{10}$, so $(r^2-7)^2 = 4\cdot 10 = 40$. so $0 = (r^2-7)^2 - 40 = r^4 - 14r^2 + 49 - 40 = r^4 - 14r^2 + 9$.

So r is a root of the polynomial $f(x) = x^4 - 14x^2 + 9$. But the Rational Roots Theorem tells us that if f has a rational root, then it must be one of the numbers $-1, 1, -3, 3, -9$, or 9 (since these are the rational numbers a/b with a dividing 9 and b dividing 1). But we can check that none of these are roots: $f(\pm 1) = 1 - 14 + 9 = -4 \neq 0$, $f(\pm 3) = 81 - 14\cdot 9 + 9 = 90 - 126 = -36 \neq 0$, and $f(\pm 9) = 81^2 - 14\cdot 81 + 9 = 64\cdot 81 + 9 > 0$. So f has no rational roots, so α, which is a root, cannot be rational.

7. Find the limit of the sequence $a_n = \dfrac{n^2 - n + 1}{3n^2 - 1}$

and prove you are right using the ϵ-N definition of the limit. [Also: show how to do this quicker using our limit theorems!]

Our limit theorems tell us that $a_n = \frac{n^2-n+1}{3n^2-1} = \frac{1-(1/n)+(1/n)^2}{3-(1/n)^2}$ has limit 1/3, since the limit of a quotient is the quotient of the limits, and, since $1/n \to 0$ as $n \to \infty$, the numerator converges to $1-0+(0)^2 = 1$, since limits behave well under sum and difference and product, while the denominator converges to $3-(0)^2$ for the same reasons.

Having found the limit we prove that it works by computing
$|a_n - \frac{1}{3}| = |\frac{n^2-n+1}{3n^2-1} - \frac{1}{3}| = |\frac{(n^2-n+1)(3)-(1)(3n^2-1)}{(3n^2-1)(3)}| = |\frac{3n^2-3n+3-3n^2+1}{(3n^2-1)(3)}| = |\frac{4-3n}{(3n^2-1)(3)}| = \frac{3n-4}{3(3n^2-1)}$,

since the numerator of $\frac{4-3n}{(3n^2-1)(3)}$ is negative for $n \geq 2$ and the denominator is positive for $n \geq 1$. This is the quantity that we wish to show can be made small ($< \epsilon$), so long as n is large enough.

But $|a_n - \frac{1}{3}| = \frac{3n-4}{3(3n^2-1)} < \frac{3n}{3(3n^2-1)} < \frac{3n}{3(3n^2-n^2)} = \frac{3n}{3(2n^2)} = \frac{1}{2n}$, since at every step we either made the numerator larger or the denominator smaller (but not negative!). So, given an $\epsilon > 0$, if we choose an $N \in \mathbb{N}$ so that $N \geq \frac{1}{\epsilon}$, then $n \geq N$ implies that $|a_n - \frac{1}{3}| < \frac{1}{2n} < \frac{1}{2N} < \frac{1}{N} < \frac{1}{1/\epsilon} = \epsilon$.

[There are many other ways that we could have done this.]

So for every $\epsilon > 0$¡ we can find an $N \in \mathbb{N}$ so that $n \geq N$ implies that $|a_n - \frac{1}{3}| < \epsilon$. SO $a_n \to \frac{1}{3}$ as $n \to \infty$.