Math 325, Section 1

Exam 2 Practice problems

1. Show that every *subsequence* $(a_{n_k})_{k=1}^{\infty}$ of a *monotonic* sequence $(a_n)_{n=1}^{\infty}$ is also monotonic.

2. Show, by example, that it is possible for a function $f : D \to \mathbb{R}$ to be continuous, for a number a to be an accumulation point of D, but the limit $\lim_{x \to a} f(x)$ does not exist.

[This problem is worded differently than we would word it this semester. Think that $D = (a, b)$ for some $b > a$.]

3. Show that if $f : [0, 2] \to \mathbb{R}$ is *continuous* and $f(0) = f(2)$, then there is a(t least one) $c \in [0, 1]$ satisfying $f(c) = f(c + 1)$.

[Hint: construct a second function that you can apply the intermediate value theorem to, to get the conclusion that we want!]

4. Show that if $A, B, C \subseteq \mathbb{R}$ and the functions $f : A \to B$ and $g : B \to C$ are both *uniformly continuous*, then the composition $g \circ f : A \to C$ [defined by $(g \circ f)(x) = g(f(x))$] is also uniformly continuous.

5. Let $(a_n)_{n=1}^{\infty}$, $(b_n)_{n=1}^{\infty}$ be bounded sequences, and suppose that, for some subsequence $(b_{n_k})_{k=1}^{\infty}$, $a_k \leq b_{n_k}$ for all k. Show that $\lim_{n \to \infty} a_n \leq \lim_{n \to \infty} b_n$ provided both limits exist!

(Hint: You *can* take the high road, and just quote theorems, but there is also another way.)

6. Suppose that $f : [0, 1] \to [0, \infty)$ is a continuous function, with the property that, for all $x \in [0, 1]$, there is a $y \in [0, 1]$ such that $f(y) \leq (1/2)f(x)$. Show that there is a $c \in [0, 1]$ such that $f(c) = 0$.

(Hint: use the hypothesis to find a sequence x_n in $[0, 1]$ such that $f(x_n) \to 0$.)

7. Show that there is no continuous function $F : \mathbb{R} \to \mathbb{R}$ satisfying $f([0, 1]) = [0, \infty)$.

8. Show that the function f:$\mathbf{R} \to \mathbf{R}$ defined by

$$f(x) = x^3 + 3x - 7$$

has exactly one root between -1 and 2 (i.e., show it has at least one root, and doesn't have two!).