Math 325, Section 1

Exam 2 Practice problems: solutions

1. Show that every *subsequence* $(a_{n_k})_{k=1}^{\infty}$ of a *monotonic* sequence $(a_n)_{n=1}^{\infty}$ is also monotonic.

Suppose first that a_n is monotone increasing, so $n \geq m$ implies that $a_n \geq a_m$. [If you take the approach that increasing means that $a_{n+1} \geq a_n$ for every $n \in \mathbb{N}$, then this statement can be established by induction on n: for $n = m$ we have $a_n = a_m \geq a_m$, and if $a_n \geq a_m$ then $a_{n+1} \geq a_n \geq a_m$¡ so $a_{n+1} \geq a_m$, giving the inductive step.]

Now suppose that a_{n_k} is a subsequence of a_n. Then $n_{k+1} > n_k$ for every k, and so since a_n is increasing we have that $a_{n_{k+1}} \geq a_{n_k}$. Then by induction we again have that $r \geq s$ implies that $a_{n_r} \geq a_{n_s}$, so a_{n_k} is monotone increasing. This establishes our result.

A symmetric argument, reversing all of the inequalities involving the sequence a_n, establishes the analogous result for monotone decreasing sequences.

2. Show, by example, that it is possible for a function $f : D \to \mathbb{R}$ to be continuous, for a number a to be an accumulation point of D, but the limit $\lim_{x\to a} f(x)$ <u>does</u> <u>not</u> <u>exist</u>.

[This problem is worded differently than we would word it this semester. Think that $D = (a, b)$ for some $b > a$.]

We wish the limit not to exist; but if $a \in D$ then continuity <u>at</u> <u>a</u> would require that $\lim_{x\to a} f(x) = f(a)$, and so in particular the limit must exist! So our example must rely on the number a not being in the domain D of our function f.

From here we can construct many examples; forcing the limit to not exist can be accomplished by making f 'blow up' as x approaches a, or oscillate wildly, or approach one value from one side and another value from the other. So, for example,

$f(x) = 1/x$, with domain $D = (0, \infty)$, has 0 as an accumulation point of D but the limit as x approach 0 does not exist, because if $1/x \to L$ as $x \to 0$, then $x = 1/(1/x) \to 1/L$ so by uniqueness of limits, $1/L = 0$, so $1 = L \cdot 0 = 0$, which is absurd. Note that f is continuous on D, since it is the reciprocal of x, whic his continuous and non-zero on D.

$g(x) = \sin(1/x)$, with domain $(0, \infty)$, has 0 as an accumulation point of D but the limit as x approach 0 does not exist, because as $x \to 0$, $1/x$ grows arbitrarily large, so $\sin(1/x)$ takes the values 1 and -1 repeatedly as $x \to 0$. Put more bluntly, $\sin(1/(1/(n+1/2)\pi)) = \sin(x_n) = 1$ and $\sin(1/(1/(n+3/2)\pi)) = \sin(y_n) = -1$, with $x_n \to 0$ and $y_n \to 0$, which violates the uniquness of limits (since $1 \neq -1$), unless $g(x)$ has no limit as $x \to 0$. Note that g is continuous on D, since it is the composition of $\sin(x)$ and the function f above.

$h(x) = x/|x|$, with domain $D = \mathbb{R} \setminus \{0\}$ is continuous, since it is -1 for $x < 0$ and 1 for $x > 0$, so for any point c in D there is a $\delta > 0$ so that h is constant (hence continuous) on $(c-\delta, c+\delta)$. But the limit of h as x approaches 0 does not exist, since there are sequences $x_n = -1/n$ and $y_n = 1/n$ so that $h(x_n) = -1 \to -1$ and $h(y_n) = 1 \to 1$, so for the limit to exist we would require $1 = -1$, which is (still) absurd.

3. Show that if $f : [0, 2] \to \mathbb{R}$ is *continuous* and $f(0) = f(2)$, then there is a(t least one) $c \in [0, 1]$ satisfying $f(c) = f(c + 1)$.

[Hint: construct a second function that you can apply the intermediate value theorem to, to get the conclusion that we want!]

The function $f_1 = f : [0,1] \to \mathbb{R}$ (i.e., with smaller domain) is continuous, as is $f_2 = f : [1,2] \to \mathbb{R}$. Also, the function $g(x) = x+1$, $g : [0,1] \to [1,2]$ is continuous (it is a polynomial!). So the function $h : [0,1] \to \mathbb{R}$ given by $h(x) = f(x) - f(x+1) = f_1(x) - f_2(g(x))$ is continuous (as the difference of two continuous functions, one of them continuous as the composition of two continuous functions).

But then $h(0) = f(0) - f(1) = \alpha$ and $h(1) = f(1) - f(2) = f(1) - f(0) = -[f(0) - f(1)] = -\alpha$. So one of three things is true: $\alpha > 0$ and so $-\alpha = f(1) \le 0 \le f(0) = \alpha$, or $\alpha >< 0$ and so $\alpha = f(0) \le 0 \le f(1) = -\alpha$, or $\alpha = 0$ and so $\alpha = f(0) \le 0 \le f(1) = -\alpha$. In every case, 0 lies between $h(0)$ and $h(1)$, and so by the Intermediate Value Theorem, there is a $c \in [0,1]$ so that $h(c) = f(c) - f(c+1) = 0$, i.e., $f(c) = f(c+1)$. This establishes our result.

4. Show that if $A, B, C \subseteq \mathbb{R}$ and the functions $f : A \to B$ and $g : B \to C$ are both *uniformly continuous*, then the composition $g \circ f : A \to C$ [defined by $(g \circ f)(x) = g(f(x))$] is <u>also</u> uniformly continuous.

Since f is uniformly continuous, for every $\eta > 0$ there is a $\delta > 0$ so that, if $x, y \in A$ and $|x - y| < \delta$, then $|f(x) - f(y)| < \eta$.

Since g is uniformly continuous, for every $\epsilon > 0$ there is an $\eta > 0$ so that, if $z, w \in B$ and $|z - w| < \eta$, then $|g(z) - g(w)| < \epsilon$.

But now suppose that $\epsilon > 0$ is given; then pick $\eta > 0$ as in the second statement, and <u>then</u> pick a $\delta > 0$ as in the first statement. Then if $x, y \in A$ and $|x - y| < \delta$, then we have $|f(x) - f(y)| < \eta$. But then $f(x), f(y) \in B$, and so we have $g(f(x)) - g(f(y))| < \epsilon$.

So we have that for every $\epsilon > 0$ there is a $\delta > 0$ so that if $x, y \in A$ and $|x - y| < \delta$, then $|(g \circ f)(x) - (g \circ f)(y)| = |g(f(x)) - g(f(y))| < \epsilon$, and so $g \circ f$ is uniformly continuous.

5. Let $(a_n)_{n=1}^\infty$, $(b_n)_{n=1}^\infty$ be bounded sequences, and suppose that, for some subsequence $(b_{n_k})_{k=1}^\infty$, $a_k \le b_{n_k}$ for all k. Show that $\lim\limits_{n\to\infty} a_n \le \lim\limits_{n\to\infty} b_n$ provided both limits exist!

(Hint: You *can* take the high road, and just quote theorems, but there is also another way.)

Let $a_n \to L$ and $b_n \to M$ as $n \to \infty$. Then since a subsequece of a convergent sequence converges to the same limit, we know that $b_{n_k} \to M$ as $k \to \infty$. But since $a_k \to L$ as $k \to \infty$ (we are just changing names) and $a_k \le b_{n_k}$, this implies that $L \le M$, since $b_{n_k} - a_k \ge 0$ implies that $b_{n_k} - a_k \to M - L \ge 0$.

That, I think, was the high road; I'm no longer sure what I thought the "other way" was....

6. Suppose that $f : [0,1] \to [0,\infty)$ is a continuous function, with the property that, for all $x \in [0,1]$, there is a $y \in [0,1]$ such that $f(y) \le (1/2)f(x)$. Show that there is a $c \in [0,1]$ such that $f(c) = 0$.

(Hint: use the hypothesis to find a sequence x_n in $[0,1]$ such that $f(x_n) \to 0$.)

If we choose any $x_1 \in [0,1]$ and set $f(x_1) = r$, then our hypothesis tells us that we can find an $x_2 \in [0,1]$ with $f(x_2) \le f(x_1)/2 = r/2$. Continuing, we can then find $x_3 \in [0,1]$ with $f(x_3) \le f(x_2)/2$, so $f(x_3) \le r/4 = r/2^2$. This suggests an induction argument:

Claim: for every $n \in \mathbb{N}$, there is an $x_n \in [0,1]$ with $f(x_n) \le r/2^{n-1}$. Proof: x_2 provides the base case. If, by induction, we have x_n, then the problem assumption yields an x with

$f(x) \le f(x_n)/2 \le (r/2^{n-1})/2 = r/2^n$, so $x_{n+1} = x$ gives our inductive step. So the claim is true by induction.

Having found $x_n \in [0,1]$ with $0 \le f(x_n) \le r/2^{n-1}$, then since $r/2^{n-1} \to 0$ as $n \to \infty$, the Squeeze Play Theorem tells us that $f(x_n) \to 0$ as $n \to \infty$.

But now $(x_n)_{n=1}^{\infty}$ is a bounded sequence, so it has a convergent subsequence $x_{n_k} \to z \in [0,1]$. Since f is continuous, we know that $f(x_{n_k}) \to f(z)$ as $k \to \infty$. But since $(f(x_{n_k}))_{k=1}^{\infty}$ is a subsequence of $(f(x_n))_{n-1}^{\infty}$, we <u>also</u> know that $f(x_{n_k}) \to 0$ as $k \to \infty$. So $f(z) = 0$ (otherwise we can use $\epsilon = |f(z) - 0| > 0$ to show that one of our limit results is false...). So: we have found a $z \in [0,1]$ with $f(z) = 0$, as desired.

7. Show that there is no continuous function $F : \mathbb{R} \to \mathbb{R}$ satisfying $f([0,1]) = [0,\infty)$.

Suppose there were such a function. Then the Extreme Value Theorem would tell us that there is a $c \in [0,1]$ so that $0 \le f(x) \le f(c)$ for every $x \in [0,1]$. But this then means that there is no x so that $f(x) = f(c) + 1$. So f cannot take on every non-negative value, so it cannot have image $[0,\infty)$.

Alternatively, we could argue more fundamentally: if there were such an F, then for every $n \in \mathbb{N}$ thre is an $x_n \in [0,1]$ so that $f(x_n) \ge n$. The sequence $(x_n)_{n=1}^{\infty}$, since it is bounded, as a convergent subsequence, x_{n_k}, converging to $z \in [0,1]$ (since $0 \le x_{n_k} \le 1$ for every k). But then continuity requires that $f(x_{n_k}) \to f(z) \in \mathbb{R}$ as $k \to \infty$, but $f(x_{n_k} \ge n_k \ge k$ implies that $f(x_{n_k}) \to \infty$. This contradiction shows that F cannot exist.

8. Show that the function f:$\mathbf{R}\to\mathbf{R}$ defined by

$$f(x) = x^3 + 3x - 7$$

has <u>exactly</u> one root between -1 and 2 (i.e., show it has at least one root, and <u>doesn't</u> have two!).

$f : [-1,2] \to \mathbb{R}$ is continuous, since f is a polynomial. In addition, $f(-1) = (-1)^3 + 3(-1) - 7 = -1 - 2 - 7 = -11 < 0$, and $f(2) = 2^3 + 3 \cdot 2 - 7 = 8 + 6 - 7 = 7 > 0$. So 0 lies between $f(-1)$ and $f(2)$, and so the intermediate value theorem tells us that $f(x) = 0$ for at least one $x \in [-1,2]$.

To show that there cannot be <u>two</u> such values of x, we can show that $x < y$ implies that $f(x) < f(y)$; that is, f is an increasing function. <u>If</u> we had derivatives, we could argue that $f'(x) = 3x^2 + 3 \ge 3 > 0$ implies that f is increasing, <u>but</u> <u>we</u> <u>don't</u>! Instead, as argue directly:

$f(y) - f(x) = (y^3 + 3y - 7) - (x_3 + 3x - 7) = (y^3 - x^3) + 3(y - x) = (y - x)(y_x^2 y_x^2) + 3(y - x) = (y - x)((y + \frac{x}{2})^2 + \frac{3}{4}x^2 + 3)$

(by completing the square), and $y - x$ is positive (by hypothesis) and $(y + \frac{x}{2})^2 + \frac{3}{4}x^2 + 3)$ is a sum of non-negative and positive numbers, so it positive. So their product, $f(y) - f(x)$, is positive, so $f(y) > f(x)$.
So if $f(x) = f(y)$ and $x \ne y$, then either $x < y$ (so $f(x) < f(y)$, a contradiction) or $y < x$ (so $f(y) < f(x)$, a contradiction!). So $x \ne y$ implies that $f(x) \ne f(y)$, so f can take the given value (like 0 (!)) at most once.