Fourier series, Parseval's Identity, and $\sum_{n=1}^{\infty} \frac{1}{n^2}$

Idea: Fourier series are a different way to express a function as a sum of 'nicer' functions. The nice functions are trig functions (instead of powers).

The other idea: the Taylor series of f is built using information from near the center $x = a$ of the series (higher derivatives at a). For many functions this tells us nothing (i.e., we get no good approximation) far from a. E.g., it can't tell us anything past a point of discontinuity of f. Fourier series use *integration* to capture information 'averaged' over an entire interval. This allows it, for example, to approximate discontinuous functions!

Start with a *periodic* function f, with period (any number is OK, like) 2π, so $f(x+2\pi) = f(x)$ for every x. The idea: express f as an (infinite) sum of nice functions, also having period 2π. A natural choice: the functions $\sin(nx)$ and $\cos(nx)$. So we attempt to write

$$f(x) = \sum_{n=0}^{\infty} a_n \sin(nx) + b_n \cos(nx)$$

Two immediate questions: does such a series converge, and can we actually do this?! Usually, yes! Just as with Taylor series, the right question to ask is: what values must a_n and b_n have? The answer is obtained by integration!

Since $\int_{-\pi}^{\pi} \sin(mx)\cos(nx)\ dx = 0$ for all m and n (the integrands are odd functions), and $\int_{-\pi}^{\pi} \sin(mx)\sin(nx)\ dx = 0$ and $\int_{-\pi}^{\pi} \cos(mx)\cos(nx)\ dx = 0$ for $m \neq n$, while $\int_{-\pi}^{\pi} \sin(nx)\sin(nx)\ dx = \int_{-\pi}^{\pi} \cos(nx)\cos(nx)\ dx = \pi$ (these can be verified by integration by parts/double angle formulas), this suggests that

$\int_{-\pi}^{\pi} f(x)\sin(mx)\ dx = \sum_{n=0}^{\infty} a_n \int_{-\pi}^{\pi} \sin(nx)\sin(mx)\ dx\ + b_n \int_{-\pi}^{\pi} \cos(nx)\sin(mx)\ dx = \pi a_m$

and

$\int_{-\pi}^{\pi} f(x)\cos(mx)\ dx = \sum_{n=0}^{\infty} a_n \int_{-\pi}^{\pi} \sin(nx)\cos(mx)\ dx\ + b_n \int_{-\pi}^{\pi} \cos(nx)\cos(mx)\ dx = \pi b_m$

and so $a_n = \dfrac{1}{\pi}\displaystyle\int_{-\pi}^{\pi} f(x)\sin(nx)\ dx$ and $b_n = \dfrac{1}{\pi}\displaystyle\int_{-\pi}^{\pi} f(x)\cos(nx)\ dx$.

So *if* we can express a function as an infinite sum of trig functions, this is what the coefficients must be equal to! For example, if we compute this for the "square wave", the function f with $f(x) = -1$ for $x \in [-\pi, 0)$ and $f(x) = 1$ for $x \in [0, \pi)$ (and which then repeats this pattern in both directions), some computations give us that

$b_n = 0$ (since f is an odd function) and $a_n = \dfrac{2(1-(-1)^n)}{n\pi}$.

$$\text{So } f(x) = \sum_{n=1}^{\infty} \frac{2(1-(-1)^n)}{n\pi}\sin(nx) = \sum_{k=0}^{\infty} \frac{4}{(2k+1)\pi}\sin((2k+1)x)\ .$$

It is somewhat beyond the scope of our course to verify this, but the theory behind all of this is that the coefficients we have computed do succeed in minimizing the integral

$$\int_{-\pi}^{\pi} [f(x) - \sum_{n=0}^{N}(a_n \sin(nx) + b_n \cos(nx))]^2\ dx$$

for every N. [If you know the words (from linear algebra), these partial sums are orthogonal projections in the vector space of functions, where the 'inner product' is $\langle f, g\rangle = \int_{-\pi}^{\pi} f(x)g(x)\ dx$. The functions $\sin(nx)$, $\cos(nx)$ are an orthogonal set of 'vectors'!] These integrals decrease with N, and (usually!) converge to 0. This implies that over <u>most</u> of the interval $[-\pi, \pi]$ the 'error' $\left|f(x) - \sum_{n=0}^{N}[a_n \sin(nx) + b_n \cos(nx)]\right|$ must be small.

Also somewhat beyond our scope, but the reason they came up in class, is **Parseval's Identity**, which relates the squares of the Fourier coefficients to the integral of the (square of the) function. The identity, for the form of the series that we have introduced, is

$$\text{If } f(x) = \sum_{n=0}^{\infty} a_n \sin(nx) + b_n \cos(nx), \qquad \text{then} \qquad \sum_{n=0}^{\infty} (a_n^2 + b_n^2) = \frac{1}{\pi} \int_{-\pi}^{\pi} (f(x))^2 \, dx \ .$$

To see why this should be so, we can write

$$\int_{-\pi}^{\pi} (f(x))^2 \, dx = \int_{-\pi}^{\pi} (\sum_{n=0}^{\infty} a_n \sin(nx) + b_n \cos(nx))^2 \, dx$$

$$= \int_{-\pi}^{\pi} (\sum_{n=0}^{\infty} a_n \sin(nx) + b_n \cos(nx))(\sum_{m=0}^{\infty} a_m \sin(mx) + b_m \cos(mx)) \, dx$$

$$= \int_{-\pi}^{\pi} \sum_{m=0}^{\infty} \sum_{n=0}^{\infty} (a_n a_m \sin(nx)\sin(mx) + b_n a_m \cos(nx)\sin(mx)$$
$$+ a_n b_m \sin(nx)\cos(mx) + b_n b_m \cos(nx)\cos(mx) \, dx$$

$$= \sum_{m=0}^{\infty} \sum_{n=0}^{\infty} \int_{-\pi}^{\pi} a_n a_m \sin(nx)\sin(mx) + b_n a_m \cos(nx)\sin(mx)$$
$$+ a_n b_m \sin(nx)\cos(mx) + b_n b_m \cos(nx)\cos(mx) \, dx$$

$$= \sum_{m=0}^{\infty} \sum_{n=0}^{\infty} \int_{-\pi}^{\pi} a_n a_m \sin(nx)\sin(mx) \, dx + \int_{-\pi}^{\pi} b_n a_m \cos(nx)\sin(mx) \, dx$$
$$+ \int_{-\pi}^{\pi} a_n b_m \sin(nx)\cos(mx) \, dx + \int_{-\pi}^{\pi} b_n b_m \cos(nx)\cos(mx) \, dx \ ,$$

where some of these steps requires some proof! (the product to two infinite sums is a doubly-infinite sum, and the integral of an infinite sum is an infinite sum of integrals; both of these can fail without some additional hypotheses!).

But if we can get past those conditions, then good things happen! Most of the integrals we have here are equal to zero, except for when $n = m$, where

$$\int_{-\pi}^{\pi} \cos(nx)\cos(nx) \, dx = \int_{-\pi}^{\pi} \sin(nx)\sin(nx) \, dx = \pi.$$

So: $\displaystyle\int_{-\pi}^{\pi} (f(x))^2 \, dx = \pi \sum_{n=0}^{\infty} (a_n^2 + b_n^2)$, as desired.

If we apply this to the square wave function above, where $b_n = 0$,

and $a_n = 0$ when n is even and $a_n = \dfrac{4}{n\pi}$ when n is odd, this yields

$$2\pi = \int_{-\pi}^{\pi} (\pm 1)^2 \, dx = \pi \sum_{k=0}^{\infty} (\frac{4}{(2k+1)\pi})^2 = \frac{16}{\pi} \sum_{k=0}^{\infty} \frac{1}{(2k+1)^2} \ , \qquad \text{so} \qquad \sum_{k=0}^{\infty} \frac{1}{(2k+1)^2} = \frac{\pi^2}{8}.$$

To recover the series that we were really after, we can either apply this same approach, using the function $f(x) = x$ on $[-\pi, \pi]$, or use a clever little trick:

$$S = \sum_{n=1}^{\infty} \frac{1}{n^2} = \sum_{n=1}^{\infty} \frac{1}{(2n)^2} + \sum_{n=1}^{\infty} \frac{1}{(2n-1)^2} = \sum_{n=1}^{\infty} \frac{1}{4n^2} + \sum_{k=0}^{\infty} \frac{1}{(2k+1)^2} = \frac{1}{4} \sum_{n=1}^{\infty} \frac{1}{n^2} + \sum_{k=0}^{\infty} \frac{1}{(2k+1)^2},$$

so $S = \dfrac{S}{4} + \dfrac{\pi^2}{8}$, so $\dfrac{3}{4}S = \dfrac{\pi^2}{8}$ and $S = \dfrac{4\pi^2}{24} = \dfrac{\pi^2}{6}$.

$$\text{So } \sum_{n=1}^{\infty} \frac{1}{n^2} = \frac{\pi^2}{6} \ .$$