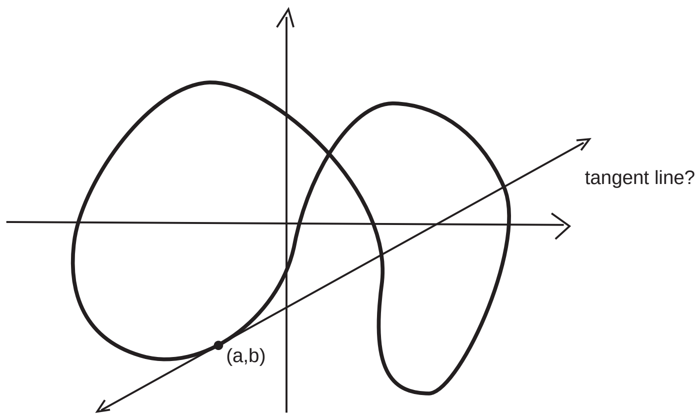
Math 106 Exam 2 Topics

Implicit differentiation

We can differentiate functions; what about *equations*? (e.g., $x^2 + y^2 = 1$)

graph <u>looks</u> <u>like</u> it has tangent lines



Idea: <u>Pretend</u> equation defines y as a function of x : $x^2 + (f(x))^2 = 1$ and differentiate!

$2x + 2f(x)f'(x) = 0$; so $\quad f'(x) = \dfrac{-x}{f(x)} = \dfrac{-x}{y}$

Different notation:

$x^2 + xy^2 - y^3 = 6$; then $2x + (y^2 + x(2y\dfrac{dy}{dx}) - 3y^2\dfrac{dy}{dx} = 0$

$$\frac{dy}{dx} = \frac{-2x - y^2}{2xy - 3y^2}$$

Application: extend the power rule

$\dfrac{d}{dx}(x^r) = rx^{r-1}$ works for any *rational* number r

($y = x^{p/q}$ means $y^q = x^p$; differentiate!)

Inverse functions and their derivatives

Basic idea: run a function backwards

y=f(x) ; 'assign' the value x to the input y ; x=g(y)

need g a function; so need f is one-to-one

f is one-to-one: if f(x)=f(y) then x=y ; if $x \neq y$ then $f(x) \neq f(y)$

$g = f^{-1}$, then $g(f(x)) = x$ and $f(g(x)) = x$ (i.e., $g \circ f$=Id and $f \circ g$=Id)

finding inverses: rewrite y=f(x) as x=some expression in y

graphs: if (a,b) on graph of f, then (b,a) on graph of f^{-1}

graph of f^{-1} is graph of f, reflected across line y=x

horizontal lines go to vertical lines; horizontal line test for inverse

derivative of the inverse: $\quad f'(f^{-1}(x)) \cdot (f^{-1})'(x) = 1$

if $f(a) = b$, then $(f^{-1})'(b) = 1/f'(a)$

$\dfrac{d}{dx}(\ln x) = 1/x$; $\dfrac{d}{dx}(\ln(f(x))) = \dfrac{f'(x)}{f(x)}$ This gives us:

Logarithmic differentiation: $f'(x) = f(x)\dfrac{d}{dx}(\ln(f(x)))$

useful for taking the derivative of products, powers, and quotients

$\ln(a^b)$ **should be** $b\ln a$, so $a^b = e^{b\ln a}$

$a^x = e^{x\ln a}$; $\dfrac{d}{dx}(a^x) = a^x \ln a$

$x^r = e^{r\ln x}$ (makes sense for *any real number* r) ; $\dfrac{d}{dx}(x^r) = e^{r\ln x}(r)(\dfrac{1}{x}) = rx^{r-1}$

Inverse trigonometric functions

Trig functions ($\sin x$, $\cos x$, $\tan x$, etc.) aren't one-to-one; make them!

$\sin x$, $-\pi/2 \le x \le \pi/2$ is one-to-one; inverse is Arcsin x

sin(Arcsin x)=x, all x; Arcsin($\sin x$)=x IF x in range above

$\tan x$, $-\pi/2 < x < \pi/2$ is one-to-one; inverse is Arctan x

tan(Arctan x)=x, all x; Arctan($\tan x$)=x IF x in range above

$\sec x$, $0 \le x < \pi/2$ and $\pi/2 < x \le \pi$, is one-to-one; inverse is Arcsec x

sec(Arcsec$x$)=x, all x; Arcsec($\sec x$)=x IF x in range above

Computing cos(Arcsin x), tan(Arcsec x), etc.; use right triangles

The other inverse trig functions aren't very useful,

they are essentially the negatives of the functions above.

Derivatives of inverse trig functions

They are the derivatives of inverse functions! Use right triangles to simplify.

$$\frac{d}{dx}(\arcsin x) = \frac{1}{\cos(\arcsin(x))} = \frac{1}{\sqrt{1-x^2}}$$

$$\frac{d}{dx}(\arctan x) = \frac{1}{\sec^2(\arctan x)} = \frac{1}{x^2+1}$$

$$\frac{d}{dx}(\text{arcsec } x) = \frac{1}{\sec(\text{arcsec } x)\tan(\text{arcsec } x)} = \frac{1}{|x|\sqrt{x^2-1}}$$

Linear approximation

Idea: The tangent line to a graph of a function makes a good approximation to the function, near the point of tangency.

Tangent line to $y = f(x)$ at $(x_0, f(x_0)$: $L(x) = f(x_0) + f'(x_0)(x - x_0)$

$f(x) \approx L(x)$ for x near x_0

Ex.: $\sqrt{27} \approx 5 + \dfrac{1}{2\cdot 5}(27-25)$, using $f(x) = \sqrt{x}$

$(1+x)^k \approx 1 + kx$, using x_0=0

$\Delta f = f(x_0 + \Delta x) - f(x_0)$, then $f(x_0 + \Delta x) \approx L(x_0 + \Delta x)$ translates to

$\Delta f \approx f'(x_0)\cdot \Delta x$

differential notation: $df = f'(x_0)dx$

So $\Delta f \approx df$, when $\Delta x = dx$ is small

In fact, $\Delta f - df$ = (diffrnce quot $-f'(x_0))\Delta x$ = (small)$\cdot$(small) = really small, goes like $(\Delta x)^2$. More precisely,

$f(x) - L(x) \approx (f''(x_0)/2)(x - x_0)^2$ when $x - x_0$ is small.

Hyperbolic functions

Any function can be written $f(x) = \frac{f(x)+f(-x)}{2} + \frac{f(x)-f(-x)}{2}$ as the sum of an <u>even</u> function $g(x) = \frac{f(x)+f(-x)}{2}$ and an <u>odd</u> function $h(x) = \frac{f(x)-f(-x)}{2}$. This can help us in understanding a function; its 'even part' and its 'odd part' each capture useful and distinct information.

Most of our basic functions are either even or odd, except for <u>exponential</u> functions. The even and odd parts of $f(x) = e^x$ are the hyperbolic functions.

$\cosh x = \frac{e^x+e^{-x}}{2}$ (even) ; $\sinh x = \frac{e^x-e^{-x}}{2}$ (odd)
These functions share much in common with trig functions!
$\cosh^2 x - \sinh^2 x = 1$
$(\cosh x)' = \sinh x$, $(\sinh x)' = \cosh x$
$\tanh x = \frac{\sinh x}{\cosh x}$; $(\tanh x)' = 1/\cosh^2 x$
$\cosh x \approx e^x$ when x is large, but is an even function; $\sinh x \approx e^x$ when x is large, but is an odd function

Applications of Derivatives

The Mean Value Theorem

You can (almost) recreate a function by knowing its derivative

Mean Value Theorem: if f is continuous on $[a,b]$ and differentiable on (a,b), then there is at least one c in (a,b) so that

$$f'(c) = \frac{f(b) - f(a)}{b - a}$$

Consequences:
Rolle's Theorem: $f(a) = f(b) = 0$; between two roots there is a critical point.
So: If a function has no critical points, it has at *most* one root!
Constant Function Theorem: A function with $f'(x)$=0 is constant.
Functions with the same derivative (on an interval) differ by a constant.

f is *increasing* on an interval if $x > y$ implies $f(x) > f(y)$
f is *decreasing* on an interval if $x > y$ implies $f(x) < f(y)$
Increasing Function Theorem: If $f'(x) > 0$ on an interval, then f is underbarincreasing on that interval. If $f'(x) < 0$ on an interval, then f is decreasing.
Racetrack Theorem: If $f(a) = g(a)$, and $f'(x) \geq g'(x)$ on $[a,b]$, then $f(x) \geq g(x)$ on $[a,b]$.

Extreme Values

The most powerful application of differential caclulus is in finding the largest and smallest values of a function.

Local extrema: near a, function values are either never higher (local max) or never lower (local min). At a local extremum, looking at difference quotients shows that either $f'(a) = 0$ or it does not exist: critical points.

So local maxs and local mins occur at critical points; how do you tell them apart?

The First Derivative Test

Near a local max, f is increasing, then decreasing; $f'(x) > 0$ to the left of the critical point, and $f'(x) < 0$ to the right.
Near a local min, the opposite is true; $f'(x) < 0$ to the left of the critical point, and $f'(x) > 0$ to the right.
If the derivative does *not* change sign as you cross a critical point, then the critical point is not a rel extremum.
Basic use: plot where a function is increasing/decreasing: First plot critical points; in between them, sign of derivative does not change.

The second derivative test

f is concave up on an interval if $f''(x) > 0$ on the interval
Means: f' is increasing; f is *bending* up.
f is concave down on an interval if $f''(x) < 0$ on the interval
Means: f' is decreasing; f is *bending* down.
A point where the concavity changes is called a point of inflection

Second derivative test: If c is a critical point and
$f''(c) > 0$, then c is a rel min (smiling!)
$f''(c) < 0$, then c is a rel max (frowning!)

Absolute Extrema

c is an (absolute) maximum for a function $f(x)$ if $f(c) \geq f(x)$ for every other x
d is an (absolute) minimum for a function $f(x)$ if $f(d) \leq f(x)$ for every other x

Extreme Value Theorem: If f is a continuous function defined on a closed interval $[a, b]$, then f actually *has* a max and a min.

Goal: figure out where they *are*!
An absolute extremum is either a local extremum or an endpoint of the interval.
A local extremum is a critical point.

So absolute extrema occur either at critical points *or* at the endpoints.
So to find the absolute max or min of a function f on an interval $[a, b]$:
(1) Take derivative, find the critical points.
(2) Evaluate f at each critical point and endpoint.
(3) Biggest value is maximum value, smallest is minimum value.

Optimization

This is really just finding the max or min of a function on an interval, with the added complication that you need to figure out *which* function, and *which* interval! Solution strategy is similar to a *related rates* problem (see below!):

Draw a picture; label things.
What do you need to maximize/minimize? Write down a formula for the quantity.
Use other information to eliminate variables, so your quantity depends on only one variable.

What (based on 'physical' constraints) is the domain of the function? Determine the largest/smallest that the variable can reasonably be (i.e., find your interval).
Turn on the max/min machine!

Families of functions and modeling

There are many *families* of functions (modifications of a single basic function) which play an important role in quantitative disciplines.

Bell curve. Basic function $f(x) = e^{-x^2}$; family of functions $f(x) = e^{-(x-a)^2/b}$

These functions represent the distribution of nearly every measurement you are likely to encounter! a is the maximum of the function (check $f'(x)$) and b changes how fast f tends to 0 as $x \to \pm\infty$. The points of inflection are at $a \pm \sqrt{b/2}$.

The logistic function. Basic function $f(x) = \frac{1}{1+e^{-t}}$; family of functions $f(x) = \frac{L}{1+Ae^{-kt}}$

These functions represent, among other things, population growth in the presence of a limiting factor. For $t << 0$ (much smaller than 0), $f(t) = \frac{L}{1+Ae^{-kt}} = \frac{Le^{kt}}{e^{kt}+A} \approx (L/A)e^{kt}$ has exponential growth, but as $t \to \infty$ we have $f(t) \to L$.
f is an increasing function (take $f'(t)$!), and has a single point of inflection at the value of t where $f(t) = L/2$. Adjusting A allows us to set the value at $t = 0$; k controls the initial rate of growth (and so determines when we reach the point of inflection). L (of course) determines the limiting value!

Ballistic motion $s(t) = -4.9t^2 + v_0t + s_0$.

These functions represent height/position of a projectile near the surface of the Earth, where accceleration is constantly $-9.8m/sec^2$. Using the Constant Function Theorem, starting with $s''(t) = -9.8$ will produce the function above, where $v_0 = s'(0)$ and $s_0 = s(0)$.

$s(t)$ reaches his maximum height when $s'(t) = v_0 - 9.8t = 0$, and reaches ground level when $s(t) = 0$. Two pieces of information (when at a specific height, velocity at specific time, maximum height, time of maximum height) are typically enough to determine v_0 and s_0, and so to completely determine $s(t)$, which in turn allows us to answer almost any other question about s !

Related Rates

Idea: If two (or more) quantities are related (a change in one value means a change in others), then their rates of change are related, too.

$xyz = 3$; pretend each is a function of t, and differentiate (implicitly).

General procedure:
Draw a picture, describing the situation; label things with variables.
Which variables, rates of change do you know, or want to know?
Find an equation relating the variables whose *rates of change* you know or want to know.
Differentiate!
Plug in the values that you know.