Math 417 Problem Set 11

Starred (*) problems are due Friday, November 30.

74. If $H, K \subseteq G$ are subgroups of G, then we can define the product (sets)

$$HK = \{hk \ : \ h \in H, \ k \in K\} \qquad \text{and} \qquad KH = \{kh \ : \ k \in K, \ h \in H\} \ .$$

Show that HK is a subgroup of $G \Leftrightarrow HK = KH$.

(*) 75. Show that if $H, K \subseteq G$ are subgroups of G, and HK is also a subgroup, then $|H| \cdot |K| = |HK| \cdot |H \cap K|$.

[Hint: show that if you pick coset representatives $A = \{a_1(H \cap K), \ldots, a_n(H \cap K)\}$ of the subgroup $H \cap K$ in H, then the map $A \times K \to HK$ given by $(a(H \cap K), k) \mapsto ak$ is a bijection.]

76. (Gallian, p.399, # 11) Show that if G is a group and G is finite and odd, then for every $g \in G$ with $g \neq e_G$ we have that g^{-1} is not conjugate to g, i.e., g^{-1} is not in $\mathrm{cl}(g)$.

[Hint: $(xgx^{-1})^{-1} = x(g^{-1})x^{-1}$ would also always be conjugate to g...]

(*) 77. If $|G| = p^n$ with p prime, show that for every k, $1 \leq k \leq n$, there is a normal subgroup $N \leq G$ with $|N| = p^k$.

[Hint: take the quotient by some element of the center of G, and use induction!]

78. Sylow's first theorem implies that the symmetric group S_5 (with $|S_5| = 5! = 2^3 \cdot 3 \cdot 5$) has a subgroup of order 8. Find one! Is the subgroup you found a normal subgroup of S_5?

[Hint: S_4 also has a subgroup of order 8. And we know a group of order 8 that acts on a certain set of 4 points...]

(*) 79. In class we showed that for p a prime, $|GL(2, \mathbb{Z}_p)| = p(p-1)(p^2-1)$. So, for example, $|GL(2, \mathbb{Z}_5)| = 5 \cdot 4 \cdot 24 = 480$, and so (by Sylow) $GL(2, \mathbb{Z}_5)$ must have elements of order 3 and of order 5. Find some! Are the subgroups that they generate normal?

80. Show that every group of order 175 is abelian.

81. According to Sylow theory, how many 5-, 7-, and 11-Sylow subgroups could a group of order $5^2 \cdot 7 \cdot 11$ have?