Math 417 Problem Set 4

Starred (*) problems are due Friday, September 21.

21. If G is a group, and $H \subseteq G$ is a subset of G so that, whenever $a, b \in H$ we have $a^{-1}b^{-1} \in H$, is this enough to guarantee that H is a subgroup of G? If yes, explain why! If not, give an example which shows that it doesn't work.

    [Hint: if $a \in H$, start listing other elements that you can guarantee are in H ...]

22. (Gallian, p.70, #34) Show that if G is a group and $H, K \subseteq G$ are subgroups of G, then their intersection $H \cap K$ is also a subgroup of G. Does this extend to the intersection of any number of subgroups of G ?

(*) 23. (Gallian, p.71, #46) Suppose that G is a group and $g \in G$ has $|g| = 5$. Show that the centralizer of g, $C(g) = C_G(g) = \{x \in G \,:\, xg = gx\}$, is equal to the centralizer of g^3, $C_G(g^3)$.

    [Hint: show that anything that commutes with g must commute with g^3, and vice versa! What, if anything, is special about the numbers 5 and 3 in this problem?]

24. (Gallian, p.73, #66) Let $G = GL_2(\mathbb{R})$ = the 2×2 invertible matrices, under matrix multiplication, and let $H = \{A \in GL_2(\mathbb{R}) \,:\, \det(A) = 2^k \text{ for some } k \in \mathbb{Z}\}$. Show that H is a subgroup of G.

(*) 25. If G is an abelian group and $n \in \mathbb{Z}$, show that $H_n = \{g \in G \,:\, g = x^n \text{ for some } x \in G\}$ (i.e., the set of n-th powers of elements of G) is a subgroup of G. Give an example where this fails if G is not abelian.

(*) 26. (Gallian, p.72, #53) Consider the element $A = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix} \in SL(2, \mathbb{Z})$ What is the order of A ? If we instead view $A = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix} \in SL(2, \mathbb{Z}_n)$ for an integer $n \geq 2$, what is the order of A ?

27. (Gallian, p.86, #15) Let G be an abelian group and let $H = \{g \in G \,:\, |g| \text{ divides } 12\}$. Prove that H is a subgroup of G. Is there anything special about 12 here? Would your proof be valid if 12 were replaced by some other positive integer? Why or why not?