Math 417 Problem Set 7

Starred (*) problems are due Friday, October 12.

43. Show that if $n \geq 3$, then $Z(S_n) = \{e\}$ is the 'trivial' subgroup of S_n.

    [Hint: Show that if $\alpha(a) \neq a$ for some a, then $\tau\alpha \neq \alpha\tau$ for some transposition τ; note that you can assume there are three distinct integers a, b, c you can build things out of (that's important! $S_2 \cong \mathbb{Z}_2$ ...).]

(*) 44. (Gallian, p.134, #44) Suppose that G is a finite *abelian* group, and that no element of G has order 2. Show that the function $\varphi : G \to G$ given by $\varphi(g) = g^2$ is an isomorphism. Show that if G is infinite then φ is a homomorphism, but need not be an isomorphism. (How many infinite abelian groups do we know at this point?)

    [Hint: show that the hypothesis about orders implies that φ is injective.]

45. (Gallian, p.135, #47) For G a group and $g \in G$ we write $\phi_g : G \to G$ to be the automorphism $\phi_g(x) = gxg^{-1}$. Show that if $\phi_g = \phi_h$ then $g^{-1}h \in Z(G)$.

46. (Gallian, p.135, #49) Show that if $n \geq 3$ and $\alpha, \beta \in S_n$ have $\phi_\alpha = \phi_\beta$, then $\alpha = \beta$ in S_n.

    [This follows quickly from two previous problems; more 'fun' would be to do it directly?]

47. Show that if $\varphi : G \to G$ is an homomorphism from G to itself, then
    $$H = \{g \in G \ : \ \varphi(g) = g\} \text{ is a subgroup of } G.$$

    [N.B.: H is called the *fixed subgroup* of φ .]

(*) 48. Let $(\mathbb{Z}[x], +, 0)$ be the group of polynomials with integer coefficients, under addition, and let $k \in \mathbb{Z}$. Show that the function $\varphi : \mathbb{Z}[x] \to \mathbb{Z}$ given by $\varphi(p(x)) = p(k)$ [the 'evaluation function'] is a homomomorphism.

(*) 49. A subgroup $H \leq G$ is called characteristic if $\varphi(H) = H$ for every $\varphi \in \text{Aut}(G)$. Show that if K is a characteristic subgroup of H and H is a characteristic subgroup of G, then K is a characteristic subgroup of G.

50. (Gallian p.209, #52) Show that if $G = \langle a \rangle$ is a cyclic group, $\phi, \psi : G \to H$ are both homomorphisms from G to H, and $\phi(a) = \psi(a)$, then $\phi = \psi$.

    ['A homomorphism from a cyclic group is completely determined by where it sends the generator.']