Math 417 Midterm Exam

Your solutions are due on your instructor's desk/in your instructor's hands by 3:00pm on Wednesday, March 15 (2016...). In developing your solutions, you may consult our text: Gallian's *Contemporary Abstract Algebra* (any edition), your class notes, any papers/solutions handed out in (our...) class, and your (Math 417) instructor. No other sources may be consulted, and if you run into any information in the normal course of your daily life which appears to be relevant to the solution of any of the problems below you should avoid following that information, to the greatest extent that your other obligations allow. Each problem letter (A,B,C,D) is worth equal credit.

Show all work! Be sure to include sufficient detail to make clear the steps you took to arrive at your answers.

A.1. Show that the group $\mathbb{Z}_{14}^*$ of units modulo 14 is a cyclic group.

A.2. Show, on the other hand, that $\mathbb{Z}_{15}^*$ is not a cyclic group. What is the largest order of any element of $\mathbb{Z}_{15}^*$?

B.1. Show that the product of two 2-cycles, $\alpha = (a_1, a_2)(b_1, b_2) \in S_n$, can be written as a product of 3-cycles. [The interesting case is $\{a_1, a_2\} \cap \{b_1, b_2\} = \emptyset$; effectively, succeeding for $\alpha = (1, 2)(3, 4)$ should show you how to do them all.] Conclude that every element of the alternating group $A_n \subseteq S_n$ is equal to a product of 3-cycles.

B.2. Express the (even) permutations $\alpha = (1, 2, 3, 4, 5, 6, 7)$ and $\beta = (1, 2, 3, 4)(5, 6, 7, 8)$ as products of 3-cycles.

C.1. Show that if $\varphi : G \to G$ is an automorphism of G, then $\varphi(Z(G)) = Z(G)$, where $Z(G)$ = the center of G.

[N.B.: Subgroups that are invariant under every automorphism of G are called 'characteristic subgroups'.]

C.2. Show that if $g, h \in G$, then $gh \in Z(G) \Leftrightarrow hg \in Z(G)$.

D. If $H, K \subseteq G$ are subgroups of G, then we can define the product (sets)

$$HK = \{hk \ : \ h \in H,\ k \in K\} \quad \text{and} \quad KH = \{kh \ : \ k \in K,\ h \in H\} .$$

Show that HK is a subgroup of $G \Leftrightarrow HK = KH$.