Math 417 Problem Set 10

Starred (*) problems are due Friday, April 22.

76. Show that if $H \triangleleft G$ and $K \triangleleft G$ are normal subgroups of the group G, then $HK = \{hk : h \in H, k \in K\}$ is a normal subgroup of G .

(*) 77. Show that if $H, K \subseteq G$ are subgroups of G, and HK is also a subgroup, then $|H| \cdot |K| = |HK| \cdot |H \cap K|$.

[Hint: show that if you pick coset representatives $A = \{a_1(H \cap K), \ldots, a_n(H \cap K)\}$ of the subgroup $H \cap K$ in H, then the map $A \times K \to HK$ given by $(a(H \cap K), k) \mapsto ak$ is a bijection.]

78. Show, using the Sylow Theorems, that a group of order 280 must have a normal Sylow subgroup.

79. According to Sylow theory, how many 5-, 7-, and 11-Sylow subgroups could a group of order $5^2 \cdot 7 \cdot 11$ have?

(*) 80. (Gallian, p.422, # 26) Show that every group of order 175 is abelian.

81. (Gallian, p.423 # 32) Show that a group of order $375 = 3 \cdot 5^3$ contains a subgroup of order 15.

82. Find a collection of distinct primes p, q, r greater than 100 for which you can show (and then show!) that every group of order pqr is isomorphic to $\mathbb{Z}_p \oplus \mathbb{Z}_q \oplus \mathbb{Z}_r \cong \mathbb{Z}_{pqr}$.

(*) 83. (Gallian, p.424, # 49) If G is a finite group and H is a normal p-Sylow subgroup of G, show that H is a characteristic subgroup of G (i.e., $\varphi(H) = H$ for every $\alpha \in \mathrm{Aut}(G)$). On the other hand, if H is not normal, show that it is not characteristic.

84. (Gallian, p.424, # 54) If G is a finite group, p is a prime, and every element of G has order p^k for some k, show that $|G| = p^n$ for some n.

[Groups with order a power of p are called *p-groups*; this problem shows that groups with elements a power of p are p-groups. (Note that Lagrange's Theorem tells us the opposite; elements of a p-group have order a power of p.)]