Math 423/823 Final Exam Practice Problems

This exam will encompass the entire range of topics presented over the semester, with a slight emphasis on the material from chapters 5, 6, and 7 (that we covered!); see the topic sheet(s). These problems are representative of the type of problem you may see on the exam, but cannot, in a few problems, cover the full range of material we have covered....

1. Find the modulus of $z = \dfrac{(1+i)(2+i)(3-2i)}{(4-i)(3i)}$.

2. Show that if $|z| = 1$, then for any complex number b we have $\left|\dfrac{z+b}{\overline{b}z+1}\right| = 1$.

3. Find the values of $z = \sqrt{1+\sqrt{i}}$. Alternative question: Using Rouché's Theorem, inside of what circle centered at 0 can we guarantee that all of the solutions to $(z^2-1)^2+1=0$ lie?

4. Show that if f is an analytic function on $0 < |z| < \infty$ and for all $z \neq 0$ we have, for some constant $A \neq 0$, $|f(z)| \geq A|z|$, then $f(z) = bz$ for some (complex) constant b) [Hint: what can you say about $g(z) = 1/f(1/z)$?]

5. Show that if f is an entire function and $f(x+2\pi) = f(x)$ for every <u>real</u> value of x, then $f(z+2\pi) = f(z)$ for every <u>complex</u> value z. [Hint: what can you say about $g(z) = f(z+2\pi) - f(z)$?]

6. Use residues to compute $\displaystyle\int_0^\infty \frac{dx}{x^6+1}$.

7. Use residues to compute $\displaystyle\int_0^\infty \frac{x^2 dx}{x^4+1}$.

8. Find the integral of $f(z) = \dfrac{z}{1+\overline{z}}$ over the line segment $\gamma(t) = t$, $0 \leq t \leq 1$.

9. Determine, for the branch of the analytic function $f(z) = z^{1/2}$ with domain all z except for $\{x+0i : x \leq 0$ and with $f(1) = 1$, whether or not $f(z_1 z_2) = f(z_1)f(z_2)$ hold for every z_1, z_2 in the domain of f. Is there a different choice of branch cut which would change the answer?

10. Write the function $f(z) = \dfrac{z}{z^2-4z+3}$ as a Laurent series which converges for $1 < |z| < 3$, and as (another!) Laurent series which converges for $3 < |z| < \infty$.

11. Compute $\displaystyle\int_0^{2\pi} \frac{\sin 4t}{\sin t}\, dt$. [Hint: convert this to an integral around the unit circle.... Note that the denominator is occasionally 0 (but then so is the top...)! Which means you will probably need to factor the bottom out of the top?]

12. Find the residue at $z = 1$ for the functions $f(z) = \dfrac{z}{z^2-1}$ and $g(z) = \dfrac{\sin(2\pi z)}{(z-1)^2}$