Math 423/823 Exam 1 Topics Covered

Complex numbers: $\mathbb{C}$: $z = x + yi$, where $i^2 = -1$; addition and multiplication 'behaves like' reals.

Formally, $x + yi \leftrightarrow (x, y)$, with $(x, y) + (a, b) = (x + a, y + b)$ and $(x, y)(a, b) = (xa - yb, ay + xb)$.

Usual properties: for $z_1, z_2, z_3 \in \mathbb{C}$:

$z_1 + z_2 = z_2 + z_1$, $z_1 z_2 = z_2 z_1$, $z_1 + (z_2 + z_3) = (z_1 + z_2) + z_3$, $z_1(z_2 z_3) = (z_1 z_2)z_3$, $z_1(z_2 + z_3) = z_1 z_2 + z_1 z_3$, $(z_1 + z_2)z_3 = z_1 z_3 + z_2 z_3$

$0 = 0 + oi = (0, 0)$, $1 = 1 + 0i + (1, 0)$: $0 + z = z$, $1 \cdot z = z$

Inverses: $(a + bi) + (-a - bi) = 0$, $(a + bi)(a - bi) = a^2 + b^2$, so
$(a + bi)^{-1} = [a/(a^2 + b^2)] - [b/(a^2 + b^2)]i$

Complex conjugates: $z = a + bi$, then $\overline{z} = z - bi$
$\overline{z + w} = \overline{z} + \overline{w}$, $\overline{zw} = \overline{z} \cdot \overline{w}$

Modulus: $|z| = |a + bi| =$ length of $(a, b) = \sqrt{a^2 + b^2}$; then $|z|^2 = z\overline{z}$, and $z^{-1} = \overline{z}/|z|^2$

Triangle inequality: $|z_1 + z_2| \leq |z_1| + |z_2|$

Quotients: $z_1/z_2 = z_1(z_2)^{-1} = (z_1\overline{z_2})/|z_2|^2$

Polar coordinates/exponential notation: $z = (x, y) = (r\cos\theta, r\sin\theta)$
$= (r\cos\theta) + i(r\sin\theta)$ =polar form of $(x, y) \in \mathbb{R}^2$, $r = |z|$,
$\theta = \arctan(y/x) = \arg(z) =$ the <u>argument</u> of z.

If $z_1 = (r_1\cos\theta_1) + i(r_1\sin\theta_1)$ $z_2 = (r_2\cos\theta_2) + i(r_2\sin\theta_2)$, then
$z_1 z_2 = (r_1 r_2\cos(\theta_1 + \theta_2)) + i(r_1 r_2\sin(\theta_1 + \theta_2))$

Suggestive notation: $e^{i\theta} = \cos\theta + i\sin\theta$, then $z = re^{i\theta}$ where $r = |z|$ and $\theta = \arg(z)$.

And $z_1 z_2 = (r_1 r_2)e^{i(\theta_1+\theta_2)}$. In part, $|z_1 z_2| = |z_1| \cdot |z_2|$.

$z = a + bi = \text{Re}(z) + i\text{Im}(z)$. $\text{Re}(z) = \dfrac{z + \overline{z}}{2}$ and $\text{Im}(z) = \dfrac{z - \overline{z}}{2i}$

Arg versus arg: $\arg(z) =$ <u>any</u> angle that z makes with the positive x-axis. $\text{Arg}(z) =$ the angle θ with $-\pi < \theta \leq \pi$ that z makes with the positive x-axis = the <u>principal</u> argument.

Under multiplication, angles add: $\arg(z_1 z_2) = \arg(z_1) + \arg(z_2)$. But Arg doesn't (always)!

For $z = re^{i\theta}$, $z^{-1} = (re^{i\theta})^{-1} = r^{-1}e^{-i\theta}$, $\overline{z} = re^{-i\theta}$, $z^n = (re^{i\theta})^n = r^n e^{in\theta}$

$re^{i\theta} = se^{i\psi}$ (with $r, s > 0$) $\Leftrightarrow r = s$ and $\theta = \psi + 2n\pi$ for some integer n

Complex roots: $w = z^{1/k}$ means $w^k = z$; for $z = re^{i\theta}$ and $w = se^{i\psi}$ this means $s = r^{1/k}$ and $\psi = \theta/k + 2n\pi/k$ for some n. Choosing $n = 0, 1, \ldots, k-1$ will produce all of the k-th roots of z.

Note that this approach essentially requires us to know how to write z in exponential form. For square roots, we can get around this using half-angle formulas:
$\cos(\theta/2) = (1/2)\sqrt{1+\cos\theta}$, $\sin(\theta/2) = (1/2)\sqrt{1-\cos\theta}$ (where $\cos\theta = x/|z|$).

Neighborhoods and open sets: $|z - w|$ =the distance from z to w. This enables us to introduce the notions of 'close' (central to the notion of limits).
$N(z_0, \epsilon) = \{z \in \mathbb{C} \ : \ |z - z_0| < \epsilon\}$ = the ϵ-neighborhood of z_0. A set $\mathcal{U} \subseteq \mathbb{C}$ is open if every point close enough to a point in $\mathcal{U}$ is also in $\mathcal{U}$: for every $z_0 \in \mathcal{U}$, there is an $\epsilon = \epsilon(z_0) > 0$ so that $N(z_0, \epsilon) \subseteq \mathcal{U}$.

Functions: A function $f : D \to \mathbb{C}$ is a assignment of a (single) number $f(z) \in \mathbb{C}$ to each $z \in D$. D=the domain of f. In general, the 'implied' domain of a function f is the largest collection of z for which the expression $f(z)$ makes sense.

We write $f(z) = f(x_y i)u + vi = u(x, y) + iv(x, y)$; $u(x, y)$ =the real part of f , $v(x, y)$ =the imaginary part of f.

Graphing complex functions: 4-tuples $((x, y), (u(x, y), v(x, y)))$ (hard to graph...) We can give a sense of what the function looks like by drawing (x, y)-planes and (u, v)-planes side by side and indicating where representative collections of points are carried by f.

(x, y)-centric: sketch the image of representative horizontal (y=const=c) and vertical (x=const=c) lines; that is, draw the curves $(u(x, c), v(x, c))$ and $(u(c, y), v(c, y))$.

polar-centric: sketch the image of circles centered at the origin (r=const=c) and rays from the origin (θ=const=α); that is, draw the curves $(u(c\cos\theta, c\sin\theta), v(c\cos\theta, c\sin\theta))$ and $(u(r\cos\alpha, r\sin\alpha), v(r\cos\alpha, r\sin\alpha))$.

(u, v)-centric: sketch the points in the (x, y)-plane which are carried to horizontal and vertical lines in the (u, v)-plane. That is, sketch the level curves of the functions $u(x, y)$ and $v(x, y)$.

In most (all?) cases, we draw both sets of curves in the same plane; points of intersection of curves represent particular values of f.

Limits: 'z is close to z_0' means $|z - z_0|$ is small. This enables us to formulate the notion of limit:
$\lim_{z \to 0} f(z) = L$ means $|f(z) - L|$ is small so long as $|z - z_0|$ is small enough (but not 0). We also write $f(z) \to L$ as $z \to z_0$. Formally:

For any $\epsilon > 0$ (our notion of 'small') there is a $\delta > 0$ (our notion of 'small enough') so that whenever $0 < |z - z_0| < \delta$ (z is 'close enough' to z_0) we must have $|f(z) - L| < \epsilon$ ($f(z)$ is 'close to' L).

The notation has been chosen to reflect the situation for functions of a real variable in part to (correctly) suggest that results from calculus that don't rely on real numbers also hold true for complex-valued functions:

Limits are unique: if we can show that $f(z) \to L$ and $f(z) \to M$ as $z \to z_0$, then $L = M$.

If in some open nbhd of z_0 we have $f(z) = g(z)$ (except possibly for $z = z_0$, then $\lim_{z \to z_0} f(z) = \lim_{z \to z_0} g(z)$

If $f(z)] \to L \neq 0$ as $z \to z_0$, then there is an $\epsilon > 0$ so that in some deleted nbhd ($0 < |z - z_0| < \delta$ for some $\delta > 0$) of z_0 we have $|f(z)| > \epsilon$. (Near a non-zero limit, a function stays away from 0.)

If $f(z) \to L \neq \infty$ as $z \to z_0$, then there is an M so that in some neighborhood of z_0 we have $|f(z)| \leq M$. (Near a finite limit, a function remains bounded.)

If $f(x + yi) = u(x, y) + iv(x, y)$, then $f(z) \to L = a + bi \Leftrightarrow u(x, y) \to a$ and $v(x, y) \to b$ as $(x, y) \to (x_0, y_0)$

If $f(z) \to L$ and $g(z) \to M$ as $z \to z_0$, then $f(z) + g(z) \to L + M$,
$f(z) - g(z) \to L - M$, $f(z)g(z) \to LM$, and (so long as $M \neq 0$) $f(z)/g(z) \to L/M$

If $g(z) \to L$ as $z \to z_0$ and $f(z) \to M$ as $z \to L$, then $f(g(z)) \to M$ as $z \to z_0$

Together with the basic building blocks: $c \to c$ and $z \to z_0$ as $z \to z_0$, we can compute many limits familiar from calculus.

Limits at ∞: For $\mathbb{C}$, infinity represents the idea of something farther away from 0 than any complex number; so 'converging to ∞' means 'having modulus grow arbitrarily large': essentially, we require that the modulus converge to ∞.

So, for example, $\lim_{z \to \infty} f(z) = L$ means that $|f(z) - L|$ is small when $|z|$ is large enough.

Most limits involving ∞ can be converted to more conventional limits using the idea that $1/z \to 0 \Leftrightarrow z \to \infty$:

$$\lim_{z \to z_0} f(z) = \infty \Leftrightarrow \lim_{z \to z_0} \frac{1}{f(z)} = 0$$

$$\lim_{z \to \infty} f(z) = L \Leftrightarrow \lim_{z \to 0} f\left(\frac{1}{z}\right) = L$$

$$\lim_{z \to \infty} f(z) = \infty \Leftrightarrow \lim_{z \to 0} \frac{1}{f\left(\frac{1}{z}\right)} = 0$$

When we wish to treat ∞ as a point in the domain or range of a function, we use the notation $\widehat{\mathbb{C}} = \mathbb{C} \cup \{\infty\}$ = the extended complex plane. E.g., $f(z) = 1/z$ can be defined as a function $f : \widehat{\mathbb{C}} \to \widehat{\mathbb{C}}$ by defining $f(0) = \infty$ and $f(\infty) = 0$.

Continuity: Just as in calculus, we often find that a limit can be computed by plugging z_0 into f. f is continuous at z_0 if $\lim_{z \to z_0} f(z) = f(z_0)$.

A function f is said to be continuous on a region R in the complex plane if f is continuous at every point of R.

Ex.: polynomials are continuous on $\mathbb{C}$; $f(z) = 1/z$ is continuous on $\mathbb{C} \setminus \{0\}$; $f(z) = \overline{z}$ is continuous on $\mathbb{C}$.

As in calculus, we have: the sum, difference, product, and (if the denominator is nonzero) quotient of continuous functions are continuous. And the composition of continuous functions is continuous.

Derivatives: again, in essence, we can borrow the notation from calculus. $f'(z_0) = \lim_{z \to z_0} \dfrac{f(z) - f(z_0)}{z - z_0}$, if the limit exists! We then say that f is differentiable at z_0.

Setting $h = z - z_0$, we can rewrite this as $f'(z_0) = \lim_{h \to 0} \dfrac{f(z_0 + h) - f(z_0)}{h}$

Unlike in calculus, we cannot really interpret this as a slope; but as in (multivariable) calculus, we interpret it in terms of linear approximation: $f(z_0 + \Delta z) \approx f(z_0) + f'(z_0)\Delta z$ for Δz small. Formally, $f(z_0 + h) = f(z_0) + f'(z_0)h + h\epsilon(h)$, with $\epsilon(h) \to 0$ as $h \to 0$.

For example, for $f(z) = z^2$ we have $f'(z) = 2z$; the calculation can be lifted from a calculus textbook. But $f(z) = \overline{z}$ is differentiable for no value of z_0. And $f(z) = |z|^2$ is differentiable only at $z_0 = 0$.

If f is differentiable at z_0, then f is continuous at z_0.

Because the definition of derivative is formally identical to the definition from calculus, our standard formulas carry through (because they don't really rely on real numbers):

$(f(z) + g(z))' = f'(z) + g'(z)$

$(f(z) - g(z))' = f'(z) - g'(z)$

$(f(z)g(z))' = f'(z)g(z) + f(z)g'(z)$

$$\left(\frac{f(z)}{g(z)}\right)' = \frac{f'(z)g(z) - f(z)g'(z)}{(g(z))^2}$$

$$\frac{d}{dz}(f(g(z)) = f'(g(z))g'(z)$$

$\dfrac{d}{dz}(z^n) = nz^{n-1}$

$\dfrac{d}{dz}(c) = 0$ for c = constant

Cauchy-Reimann Equations: Typically for a complex-valued function $f(x+iy) = u(x,y)+iv(x,y)$, the functions u and v do not make differentiable functions of z. It is the combination which is differentiable. By computing the limit as $h \to 0$ along the x-axis and y-axis separately, and equating the results, we find that if f is differentiable at $z_0 = x_0 + y_0 i$, then

$$\frac{\partial u}{\partial x} = \frac{\partial v}{\partial y} \qquad \text{and} \qquad \frac{\partial u}{\partial y} = -\frac{\partial v}{\partial x}$$

at (x_0, y_0). And in that case, $f'(z_0) = u_x(x_0,y_0) + iv_x(x_0,y_0) = v_y(x_0,y_0) - iu_y(x_0,y_0)$

This works (sort of) in reverse, as well: if u_x, u_y, v_x, and v_y are all continuous in a nbhd of (x_0, y_0), and the CR Eqns $u_x = v_y$, $u_y = -v_x$ hold at (x_0, y_0), then $f = u + vi$ is differentiable at (x_0, y_0). This allows us to compute derivatives of many of our standard 'elementary' functions.

$f(z) = e^z = e^{x+iy} = e^x e^{iy} = e^x(\cos y + i \sin y) = (e^x \cos x) + i(e^x \sin y)$, so
$u(x,y) = e^x \cos y$ and $v(x,y) = e^x \sin y$. We can check that the CR equations hold at every point, and all of the partial derivatives are continuous, so f is differentiable and
$f'(z) = u_x + iv_x = (e^x \cos x) + i(e^x \sin y) = e^z$!

Trig functions: $\cos z = \cos(x+yi) = \dfrac{e^{iz} + e^{-iz}}{2} = \cos x \cosh y - i \sin x \sinh y$

$\sin z = \sin(x+yi) = \dfrac{e^{iz} - e^{-iz}}{2i} = \sin x \cosh y + i \cos x \sinh y$
(Note the similarity to the angle sum formulas for sine and cosine!)

As a result, $(\sin z)' = \cos z$, $(\cos z)' = -\sin z$

$\sin^2 z + \cos^2 z = 1$ still holds true! We can also define, in the usual way, $\tan(z), \cot(z), \sec(z)$, and $\csc(z)$, and the differentiation rules will demonstrate that they have the expected derivatives.

A function $w = f(z)$ is analytic at a point z_0 if it is differentiable at every point in a (small) nbhd of z_0. f is analytic on a domain D if it is analytic at every point in D. (Other names are regular or holomorphic.) A function which is analytic on the entire complex plane $\mathbb{C}$ is called entire. For example, every polynomial function is an entire function. A rational function is analytic everywhere except at the roots of its denominator.

Our differentiation formulas imply that the sum, difference, product, and (except at the roots of its denominator) quotient of analytic functions are analytic. The chain rule implies that the composition of analytic functions is analytic (on the appropriately chosen domain).

An important property from calculus carries over: if f is analytic on D and $f'(z) = 0$ for every z in D, and if every pair of points in D can be joined by a (continuous) path that stays in D, then f is a constant function.

If $w = f(z)$ and $w = \overline{f(z)}$ are both analytic on a domain D (with the same condition of paths for D above), then f is a constant function. [The CR equations imply that $f' = 0$.] This in turn implies that if $w = f(z)$ is analytic and takes only real values, then f is constant.

If $w = f(z)$ is analytic on D (same path condition!) and $|f(z)|$ is constant, then f is constant.

Harmonic conjugates: If $f(z) = u(x,y) + iv(x,y)$ is analytic, then the CR equations hold. But even more, u and v have partial derivative to all orders and, as a result, $u_{xx} + u_{yy} = 0$ and $v_{xx} + v_{yy} = 0$. Such functions are called harmonic; they are solutions to Laplace's equation. Harmonic functions play an important role in many (all?) science and engineering fields.

As a result, any real-valued function that is not harmonic (e.g, $u = x^2 + y^2$) cannot be the real (or imaginary) part of an analytic function. Even more, in general a pair of harmonic functions u, v don't need to pair up to give an analytic function $f = u + iv$; the CR equations must also be satisfied. In fact, the CR equations essentially give a method for building, from a harmonic u, the (essentially unique) harmonic fcn v so that $u + iv$ is analytic. [v is called the harmonic conjugate of u]. The basic idea is that the CR equations tell us what the gradient $\nabla v = (v_x, v_y) = (-u_y, u_x)$ must be, and the harmonicity of u tells us that v_{xy} will equal v_{yx}, so the potential function v will exist, and can be recovered by integration. $v(x,y) = \int -u_y(x,y)\, dx \ + g(y)$; differentiating this w.r.t y and equating it with $v_y = u_x$ allows us to determine g (up to an additive constant).

Logarithms: $f(z) = e^z$ is not a one-to-one function ($e^{2\pi i} = 1 = e^0$), so doesn't have an inverse in the usual sense, but it does have a multi-valued inverse $g(z) = \log(z)$. [Apparently in complex variables, logarithms to other bases just aren't as popular...] We can find an expression for it by writing $w = u + vi = e^z = e^{x+yi} = (e^x \cos y) + i(e^x \sin y)$ and expressing x and y in terms of u and v:
$u^2 + v^2 = (e^x)^2$, so $x = \ln(\sqrt{u^2 + v^2}) = \ln|w|$
$\tan(y) = v/u$, so $y = \arctan(v/u) = \arg(w)$

So $\log(w) = \ln|w| + i\arg(w)$. Writing this as $g(z) = \log(z) = \frac{1}{2}\ln(x^2+y^2) + i\arctan(y/x) = u + iv$, the CR equations are satisfied, and so

$$g'(z) = u_x + iv_x = \frac{x}{x^2+y^2} + i\frac{-y}{x^2+y^2} = \frac{\overline{z}}{|z|^2} = \frac{1}{z} \ !$$

Branches: $\arg(z)$ is multi-valued; we can add multiples of 2π. So $\log(z)$ is multi-valued, too. If we want a (single-valued) function, we must restrict the values of $\arg(z)$. If we make the 'standard' choice ($pi < \mathrm{Arg}(z) \leq \pi$), we get the principal branch of the logarithm: $\mathrm{Log}(z) = \ln|z| + i\mathrm{Arg}(z)$. This function is analytic on the complex plane with the (origin and) the negative x-axis removed.
We could just as well define $\arg(z)$ (and so $\log(z)$) by insisting that it take values in $(\alpha, \alpha + 2\pi]$; then $\log(z)$ is analytic in the plane with the ray $\arg(z) = \alpha$ removed.

Because of these ambiguities, some of the familiar properties of logarithms fail to hold (when we insist that it be single-valued). For example,
$\text{Log}(i) = \pi i/2$, but $\text{Log}(i^3) = \text{Log}(i) = -\pi i/2$, which is different from $3\text{Log}(i) = 3\pi i/2$. So we cannot expect that $\log(a^b)$ and $b\log(a)$ will always be equal to one another.

However, if we are willing to think of $\log(z)$ as a multi-valued function, then the familiar identities will work: since $\text{Arg}(z_1z_2)$ and $\text{Arg}(z_1)+\text{Arg}(z_2)$ will always differ by a multiple of 2π, if we interpret $\arg(z_1)+\arg(z_2)$ as the collection of all possible sums of the multiple values of the summands, then
$\arg(z_1z_2)$ and $\arg(z_1)+\arg(z_2)$ is true. applying this same approach to the multiple values of $\log(z)$, we then find that
$\log(z_1z_2) = \log(z_1) + \log(z_2)$ and $\log(z^r) = r\log(z)$ (for r a rational number)
hold true as multi-valued expressions.

Complex exponentiation: With the exponential and logarithm functions, $\exp(z) = e^z$ and $\log(z)$, we can follow the practice from calculus to define exponentials in general: $a^b = \exp(\log(a^b)) = \exp(b\log(a))$ So for example, $z^c = \exp(c\log(z))$ can be treated as a single-valued function once a branch ($\alpha < \arg(z) \leq \alpha + 2\pi$) of $\arg(z)$ is chosen, and then the chain rule can be used to show that $(z^c)' = cz^{c-1}$ (in particular, $f(z) = z^c$ is analytic off of the ray $\arg(z) = \alpha$). Similarly, for any $a \neq 0$ we can define $a^z = \exp(z\log(a))$, and , again, choosing a particular value for $\log(a)$ makes this a single valued function, whose derivative, by the chain rule, is $a^z\log(a)$.

Inverse Trig functions: these again will be multi-valued functions, but by exploiting some complex variables we can give actual formulas for them!
Since $w = \sin(z) = \dfrac{e^{iz} - e^{-iz}}{2i}$, we have $(e^{iz})^2 - 2iw(e^{iz}) - 1 = 0$. Solving this (quadratic!) equation for e^{iz} and taking log's, we get $iz = \log(iw \pm \sqrt{1-w^2})$, so (since $\sqrt{\text{blah}}$ already has the ambiguity of sign)
$z = \arcsin(w) = -i\log(iw + \sqrt{1-w^2})$, where this is treated as a multivalued function. Choosing the principal logarithm (and the principal branch of the square root function) yields one choice of single-valued function, $\text{Arcsin}(w)$.

Taking the derivative of this expression yields (with some work) the usual formula
$(\arcsin(z))' = (1-z^2)^{-1/2}$

Similarly, $w = \tan z = \dfrac{\sin z}{\cos z}$ yields $z = \dfrac{i}{2}\log\left(\dfrac{i-w}{i+w}\right)$, and so
$\arctan z = \dfrac{i}{2}\log\left(\dfrac{i-z}{i+z}\right)$, with derivative $(\arctan z)' = \dfrac{1}{1+z^2}$!

Things we know how to do:

add, subtract, multiply, divide complex numbers

convert to/from exponential notation
compute the modulus and argument of a complex number

compute roots of complex numbers

sketch the 'graph' of a complex-valued function using coordinate lines or level curves

compute limits of functions ; show limits do not exist (approach z_0 from different directions)

compute derivatives of functions

determine (non-)differentiability using limits
determine differentiability using the Cauchy-Riemann equations

compute derivatives using the Cauchy-Riemann equations

find the harmonic conjugate of a function