Math 423/823 Exam 2 Topics Covered

Don't forget the topics from the first exam!

Logarithms: $f(z) = e^z$ is not a one-to-one function ($e^{2\pi i} = 1 = e^0$), so doesn't have an inverse in the usual sense, but it does have a multi-valued inverse $g(z) = \log(z)$. [Apparently in complex variables, logarithms to other bases just aren't as popular...] We can find an expression for it by writing $w = u + vi = e^z = e^{x+yi} = (e^x \cos y) + i(e^x \sin y)$ and expressing x and y in terms of u and v:

$u^2 + v^2 = (e^x)^2$, so $x = \ln(\sqrt{u^2 + v^2}) = \ln |w|$

$\tan(y) = v/u$, so $y = \arctan(v/u) = \arg(w)$

So $\log(w) = \ln |w| + i \arg(w)$. Writing this as $g(z) = \log(z) = \frac{1}{2} \ln(x^2 + y^2) + i \arctan(y/x) = u + iv$, the CR equations are satisfied, and so

$g'(z) = u_x + iv_x = \dfrac{x}{x^2 + y^2} + i\dfrac{-y}{x^2 + y^2} = \dfrac{\overline{z}}{|z|^2} = \dfrac{1}{z}$!

Branches: $\arg(z)$ is multi-valued; we can add multiples of 2π. So $\log(z)$ is multi-valued, too. If we want a (single-valued) function, we must restrict the values of $\arg(z)$. If we make the 'standard' choice ($\pi < \text{Arg}(z) \leq \pi$), we get the <u>principal</u> <u>branch</u> of the logarithm:

$\text{Log}(z) = \ln |z| + i\text{Arg}(z)$. This function is analytic on the complex plane with the (origin and) the negative x-axis removed.

We could just as well define $\arg(z)$ (and so $\log(z)$) by insisting that it take values in $(\alpha, \alpha + 2\pi]$; then $\log(z)$ is analytic in the plane with the ray $\arg(z) = \alpha$ removed.

Because of these ambiguities, some of the familiar properties of logarithms fail to hold (when we insist that it be single-valued). For example,

$\text{Log}(i) = \pi i/2$, but $\text{Log}(i^3) = \text{Log}(i) = -\pi i/2$, which is different from $3\text{Log}(i) = 3\pi i/2$. So we cannot expect that $\log(a^b)$ and $b \log(a)$ will always be equal to one another.
However, if we are willing to think of $\log(z)$ as a <u>multi</u>-<u>valued</u> function, then the familiar identities will work: since $\text{Arg}(z_1 z_2)$ and $\text{Arg}(z_1) + \text{Arg}(z_2)$ will always differ by a multiple of 2π, if we interpret $\arg(z_1) + \arg(z_2)$ as the collection of all possible sums of the multiple values of the summands, then

$\arg(z_1 z_2)$ and $\arg(z_1) + \arg(z_2)$ is true. applying this same approach to the multiple values of $\log(z)$, we then find that

$\log(z_1 z_2) = \log(z_1) + \log(z_2)$ and $\log(z^r) = r \log(z)$ (for r a rational number)

hold true as multi-valued expressions.

Complex exponentiation: With the exponential and logarithm functions, $\exp(z) = e^z$ and $\log(z)$, we can follow the practice from calculus to define exponentials in general: $a^b = \exp(\log(a^b)) = \exp(b \log(a))$ So for example, $z^c = \exp(c \log(z))$ can be treated as a single-valued function once a branch ($\alpha < \arg(z) \leq \alpha + 2\pi$) of $\arg(z)$ is chosen, and then the chain rule can be used to show that $(z^c)' = cz^{c-1}$ (in particular, $f(z) = z^c$ is analytic off of the ray $\arg(z) = \alpha$). Similarly, for any $a \neq 0$ we can define $a^z = \exp(z \log(a))$, and , again, choosing a <u>particular</u> value for $\log(a)$ makes this a single valued function, whose derivative, by the chain rule, is $a^z \log(a)$.

Inverse Trig functions: these again will be multi-valued functions, but by exploiting some complex variables we can give actual formulas for them!

Since $w = \sin(z) = \dfrac{e^{iz} - e^{-iz}}{2i}$, we have $(e^{iz})^2 - 2iw(e^{iz}) - 1 = 0$. Solving this (quadratic!) equation for e^{iz} and taking log's, we get $iz = \log(iw \pm \sqrt{1-w^2})$, so (since $\sqrt{\text{blah}}$ already has the ambiguity of sign)

$z = \arcsin(w) = -i\log(iw + \sqrt{1-w^2})$, where this is treated as a multivalued function. Choosing the principal logarithm (and the principal branch of the square root function) yields one choice of single-valued function, $\text{Arcsin}(w)$.

Taking the derivative of this expression yields (with some work) the usual formula

$(\arcsin(z))' = (1-z^2)^{-1/2}$

Similarly, $w = \tan z = \dfrac{\sin z}{\cos z}$ yields $z = \dfrac{i}{2}\log\left(\dfrac{i-w}{i+w}\right)$, and so

$\arctan z = \dfrac{i}{2}\log\left(\dfrac{i-z}{i+z}\right)$, with derivative $(\arctan z)' = \dfrac{1}{1+z^2}$!

Integration: Modelled on line integrals (integrate vector field around a parametrized curve), except instead of dot product we will use complex multiplication.

Complex-valued function of a real variable $\gamma(t) = x(t) + iy(t)$; think of (sometimes!) as a parametrized curve running around in the complex plane.

Derivative: $\gamma'(t) = x'(t) + iy'(t)$

Usual differentiation rules work: sum, difference, **product**, quotient

Chain Rule: $f(x+iy) = u + iv$, $\gamma(t) = x(t) + iy(t)$, then $[f(\gamma(t))]' = f'(\gamma(t)) \cdot \gamma'(t)$.

Integral (of a complex-valued function of a real variable): $f(t) = x(t) + iy(t)$

$$\int_a^b f(t)\ dt = \int_a^b x(t)\ dt + i\int_a^b y(t)\ dt$$

Usual integration rules work: constant multiple, sum, difference, u-substitution, integration by parts

Contours = parametrized curves in the complex plane: $z(t) = x(t) + iy(t)$

continuous: both $x(t)$ and $y(t)$ are continuous
differentiable: both $x(t)$ and $y(t)$ are differentiable

Typically: insist that contour is continuous, and differentiable except possibly at a finite number of points ("piecewise differentiable")

Simple path = path that never visits the same point in the plane twice (except that maybe the two endpoints agree)

Reparametrization: don't change where the path goes, just when! $f(\phi(t))$, where ϕ is either always increasing or always decreasing.

By the chain rule, $[f(\phi(t))]' = f\prime(\phi(t)) \cdot \phi'(t)$

Arclength: $|\gamma'(t)|$ = size of rate of change = 'speed'; $\int_a^b |\gamma'(t)|\ dt$ = length of the curve γ

Arclength is unchanged under reparametrization (why? u-subs)

Closed curve = parametrized curve $\gamma : [a,b] \to \mathbb{C}$ with $\gamma(a) = \gamma(b)$

Jordan Curve Theorem: Every *simple* closed curve γ separates the complex plane $\mathbb{C}$ into two regions R and S, exactly one of which is *bounded* (= for some R, every point lies within R of the origin). γ is called *positively oriented* if as we traverse γ (with increasing t) its bounded region always lies to our left. (This is also the 'counterclockwise' orientation around γ.)

Contour Integrals: For a complex-valued function $w = f(z)$ and a contour $z = \gamma(t)$, $a \le t \le b$, the contaour integral of f along γ is defined as

$$\int_\gamma f(z)\ dz = \int_a^b f(\gamma(t)\gamma'(t)\ dt$$

(if the limit exists). This integral was designed so that the number is unchanged under (orientation-preserving, i.e., $\phi'(t) > 0$) reparametrization of γ. [It picks up a minus sign (-) under orientation-reversing reparametrization.]

This integral satisfies the usual properties: behaves as expected under constant multiple, sums, differences; if the contour is split into pieces, the integral is the sum of the integrals over the pieces.

Motivating example: for $\gamma(t) = e^{it}$, $0 \le t \le 2\pi$, $\int_\gamma \frac{dz}{z} = 2\pi i$

The idea: $f(z) = 1/z$ has an antiderivative $F(z) = \text{Log}(z)$, except that this function is not defined on the negative x-axis. The failure of the antiderivative to be well-defined on all of the curve contributes to the non-zero result.

Fundamental Theorem of Complex Calculus: If $w = F(z)$ is analytic on a domain D, $F'(z) = f(z)$ on D, and $\gamma : [a,b] \to D$ is a contour that lies entirely in D, then

$$\int_\gamma f(z)\ dz = F(\gamma(b)) - F(\gamma(a))$$

In particular, the value of the integral depends only on the endpoints, and not on the particular curve used to join those endpoints. Note that this is really the same as saying that $\int_\gamma f(z)\ dz = 0$ for γ any closed path.

A basic inequality: $\left|\int_a^b z(t)\ dt\right| \le \int_a^b |z(t)| dt$

So: If $|f(z)| \le M$ along a curve γ of length L, then $\left|\int_\gamma f(z)\ dz\right| \le LM$

Application: if $w = f(z)$ satisfies $|f(re^{i\theta}| \le M(r)$ for all θ, for some function $M(r)$, and $rM(r) \to 0$ as $r \to \infty$, then for the curves $\gamma_r(t) = re^{it}$, $\theta_0 \le t \le \theta_1$, $\int_{\gamma_r} f(z)\ dz \to 0$ as $r \to \infty$.

The 'second' FTC, i.e., its converse!: If D is an open (connected) domain and f is a function so that for any curve $\gamma : [a,b] \to D$ in D, $\int_\gamma f(z)\ dz$ depends only on $\gamma(b)$ and $\gamma(a)$, then f has an antiderivative F. It can be given (choosing a 'base' point $z_0 \in D$) as:

$$F(z) = \int_\gamma f(w)\ dw \text{ where } \gamma \text{ is any curve from } z_0 \text{ to } z$$

So having an antiderivative on D is the same as having integral 0 around any closed curve in D.

Cauchy-Goursat Theorem: If $w = f(z)$ is analytic on and inside of the simple closed curve γ, then $\int_\gamma f(z)\ dz = 0$.

[Cauchy assumed $f'(z)$ is continuous, in order to use Green's Theorem!] This implies that if you are analytic, then you have an antiderivative.

Analyticity on and inside of the curve γ can be obtained 'for free' if f is analytic on a *simply-connected domain* D and γ lies entirely in D. D is *simply connected* if it is connected (any two points in D can be joined by a path in D) and for every simple closed curve [notation: scc] γ in D the bounded region guaranteed by the Jordan curve theorem lies in D. A domain which is not simply-connected is (unfortunately) called *multiply connected*.

Example: the complex plane $\mathbb{C}$ is simply-connected, as are the upper half-plane $\{z = x+iy : y > 0\}$ and the points lying off of any one ray $\{re^{ia} : r \geq 0\}$ (for a fixed $a \in \mathbb{R}$). So, e.g., for any entire function f (think: $z^2, e^z, \sin z$, etc.) and any simple closed curve γ, $\int_\gamma f(z)\ dz = 0$.

If f is analytic on γ but not at every point inside of γ (which we assume is positively oriented), then is we surround the points of non-analyticity inside of γ with positively-oriented (and disjoint) simple loops $\gamma_1, \ldots, \gamma_n$, so that f is analytic at every point that is inside of γ <u>and</u> outside of every one of the γ_i, then

$$\int_\gamma f(z)\ dz = \int_{\gamma_1} f(z)\ dz + \cdots + \int_{\gamma_n} f(z)\ dz$$

(This followed from Cauchy-Goursat by stitching together all of these loops into a single 'almost' simple loop.)

So, e.g., if δ lies inside of γ (i.e., lies in the region it bounds), and f is analyic at every point between the two scc's [and both are positively or negatively oriented], then $\displaystyle\int_\gamma f(z)\ dz = \int_\delta f(z)\ dz$. We often use this to trade an 'ugly' loop for something more standard (usually $\delta(t) = z_0 + r_0 e^{it}$ for fixed z_0 and r_0).

Cauchy Integral Formula: If f is analytic on and inside of a positively oriented scc γ, then

$$\frac{1}{2\pi i}\int_\gamma \frac{f(z)}{z - z_0}\ dz = f(z_0)$$

This has far-reaching consequences. It says that knowing the values of f along γ (and knowing that it is analytic inside of γ) is enough to be able to compute the value of f at every point inside of γ. It also gives us the tools to compute a wide range of integrals that are otherwise out of reach. It <u>also</u> gives us the tool to show the results below!

If f is analytic on and inside of (pos. or'd) γ, then $\frac{n!}{2\pi i}\int_\gamma \frac{f(z)}{(z-z_0)^{n+1}}\,dz = f^{(n)}(z_0)$.

In part, f is (infinitely) differentiable! so $f'(z)$ is analytic! But since $f = u + iv$ has $f'(z) = u_x + iv_x = v_y - iu_y$, this implies that u_x, v_x, u_y, v_y are all differentiable (and continuous). [These are facts we used before; now we have justified them.] Also, if you have an antiderivative, then you are analytic. So analytic functions are precisely the functions that have antiderivatives!

The same demonstration used for CIF shows that if γ is a simple closed curve and g is a function defined on γ, then for D the region lying inside of γ and the function $f : D \to \mathbb{C}$ defined by

$$f(z) = \frac{1}{2\pi i}\int_\gamma \frac{g(s)}{s-z}\,dz$$

is an analytic function on D ! It's derivative is $f'(z) = \frac{1}{2\pi i}\int_\gamma \frac{g(s)}{(s-z)^2}\,dz$ ...

Cauchy's Inequality: If F is analytic on and inside of a circle $C_R(t) = z_0 + Re^{it}$, and M_R is the maximum of F on the circle C_R, then $|f^{(n)}(z_0)| \leq \frac{n!M_R}{R^n}$.

Liouville's Theorem: If f is an entire function and for some M we have $|f(z)| \leq M$ for all z, then f is constant.

Fundamental Theorem of Algebra: Every polynomial with complex coefficients, of degree greater than 0, has a complex root.

Why? Otherwise $g(z) = [f(z)]^{-1}$ is a non-constant bounded entire function! The FTA implies that every polynomial completely factors as a product of linear polynomials. Over the reals, this means that every polynomial with real coefficients can be written as a product of linear and irreducible quadratic factors.

Some definite integral computations: Using $\sin t = \frac{e^{it} - e^{-it}}{2i}$ and $\cos t = \frac{e^{it} + e^{-it}}{2}$, we can write integrals $\int_0^{2\pi} F(\sin t, \cos t)\,dt$ as contour integrals around the unit circle $\gamma(t) = e^{it}$. Essentially, using the $z = e^{it}$, $dz = ie^{it}\,dt$, so $dt = \frac{dz}{iz}$, and $\sin t = \frac{z - z^{-1}}{2i}$ and $\cos t = \frac{z + z^{-1}}{2}$. So

$$\int_0^{2\pi} F(\sin t, \cos t)\,dt = \int_\gamma F\Big(\frac{z - z^{-1}}{2i}, \frac{z + z^{-1}}{2}\Big)\,\frac{dz}{iz}$$

By expressing the integrand as a rational function in z and finding the roots of the denominator which lie inside of the curve γ, we can (often) apply the Cauchy Integral Formula to evaluate the integral. For example, we found that

$$\int_0^{2\pi} \frac{dt}{a + \sin t} = \frac{2\pi}{\sqrt{a^2 - 1}} \text{ for any } a > 1$$

Maximum Modulus Principle: If D is an open domain and f is a non-constant analytic function on D, then $|f(z)|$ never achieves a maximum value at any point in D.

As this is usually uysed, it is interpreted to as that if f is analytic on and inside of a scc γ (or between a collection of curves γ_i), then $|f(z)|$ (which is continuous!) must achieve is maximum on underbarthe underbarcurve γ (or on one of the curves γ_i).

As an application, for any function $u(x, y)$ that is harmonic on a domain D, building its harmonic conjugate v, and the analytic function $f(x + iy) = e^{u(x,y)+iv(x,y)}$, the MMP implies that u must achieve its maximum only on a boundary curve of D (or not at all...).

Things we know how to do:

establish (or show the failure of) identities involving logarithms, trig functions, etc.

compute solutions to $f(z) = a$ for f a logarithm, trig function, exponential, etc.

compute derivatives involving logarithms, exponentials, etc.

compute integral of complex-valued function of real variable

use integrals of complex-valued functions to find integrals of real-valued functions

compute contour integrals by converting to the above

compute contour integrals using Cauchy-Goursat and Cauchy Integral Formula

Use Cauchy integral formula to obtain information about derivatives of analytic function

compute integrals of real-valued functions by expressing as real/imaginary part of contour integral

use Cauchy's Inequality to obtain information about an analytic function (model: Liouville's Theorem)