Math 423/823 Topics Covered Since Exam 2

Don't forget the topics from the first two exams!

Power Series: Our basic guiding principle is that everything behaves the way it does for calculus/real variables, only more so!

$\sum_{n=0}^{\infty} a_n$ converges to L if the partial sums $S_N = \sum_{n=0}^{N} a_n$ are eventually as close to L as we would ever need them to be: $|S_N - L| \to 0$ as $N \to \infty$.

$\sum_{n=0}^{\infty}(a_n + b_n) = \sum_{n=0}^{\infty} a_n + \sum_{n=0}^{\infty} b_n$ hskip.2in $\sum_{n=0}^{\infty} ca_n = c\sum_{n=0}^{\infty} a_n$

Power Series: $f(z) = \sum_{n=0}^{\infty} a_n z^n$; domain = the z for which it converges!

Our basic workhorse series: $\sum_{n=0}^{\infty} z^n = \frac{1}{1-z}$; converges for $|z| < 1$, diverges for $|z| > 1$.

The big theorem: If $w = f(z)$ is analytic at $z = z_0$ (so there is a largest R so that f is analytic for all z with $|z - z_0| < R$ [possibly $R = \infty$], then the power series

$$\sum_{n=0}^{\infty} \frac{f^{(n)}(z_0)}{n!}(z - z_0)^n$$

converges for all $|z - z_0| < R$ and equals $f(z)$ there.

So, e.g., $f(z) = e^z$ is an entire function, and $f^{(n)}(0) = 1$ for all n, so e^z equals the power series you remember from calculus, for all complex numbers z. The same is true for the familiar power series representations of $\sin z, \cos z$, etc.

Prop: If $f(z) = \sum_{n=0}^{\infty} a_n(z - z_0)^n$ converges for some $z = z_1$, then setting $R = |z_1 - z_0|$, f converges for all $|z - z_0| < R$.

So if $f(z) = \sum_{n=0}^{\infty} a_n(z - z_0)^n$ does not converge for some $z = z_2$ (and $r = |z_2 - z_0|$), then f does not converge for any z with $|z - z_0| > r$, because if it did converge, it would have to converge at $z = z_2$, too. The closest point z_1 to z_0 for which f does not converge defines the *circle of convergence* $|z - z_0| = R = |z_1 - z_0|$; inside of this circle, the series converges; outside of the circle, it diverges.

Laurent series: Even when a function f fails to be analytic at a point $z = z_0$, if it is analytic in a *deleted* neighborhood $0 < |z - z_0| < R$ around z_0 we can still represent it as a series centered at z_0. We 'just' need to allow for negative exponents! A power series with both negative and (possibly) positive exponents is called a *Laurent series.*

Again, our basic workhorse for doing this is $f(z) = \frac{1}{1-z}$; for $|z| > 1$,

$$\frac{1}{1-z} = -\frac{1/z}{1-1/z} = -\frac{1}{z}\sum_{n=0}^{\infty}\left(\frac{1}{z}\right)^n = \sum_{n=-1}^{\infty}(-1)z^{-n} = \sum_{n=-\infty}^{-1}(-1)z^n,$$

since $|1/z| < 1$. Based on this, we can show:

If $w = f(z)$ is analytic on a ring $R_1 < |z - z_0| < r_2$ centered at $z = z_0$, then f can be expressed as a Laurent series

$$f(z) = \sum_{n=0}^{\infty} a_n(z-z_0)^n + \sum_{n=1}^{\infty} b_n(z-z_0)^{-n} = \sum_{n=-\infty}^{\infty} c_n(z-z_0)^n,$$

where, for C = any simple closed curve (oriented counterclockwise) lying in the ring and containing z_0 in its bounded complementary region, we have

$$c_n = \frac{1}{2\pi i}\int_C \frac{f(z)}{(z-z_0)^{n+1}}\, dz$$

Note that if f is actually analytic on all of $|z - z_0| < R_2$, then for $n \le -1$, $\dfrac{f(z)}{(z-z_0)^{n+1}} = f(z)(z-z_0)^{-(n+1)}$ is also analytic (the exponent is non-negative), so $c_n = 0$ by the Cauchy-Goursat Theorem. So all of the coefficients of negative powers are zero.

Just as in calculus, inside of its circle(s) of convergence a power series $f(z) = \displaystyle\sum_{n=-\infty}^{\infty} c_n(z-z_0)^n$ is continuous, is differentiable, and its derivative can be obtained by term-by-term differentiation of its Taylor/Laurent series. This implies that a power series is analytic inside of its circle(s) of convergence. It then has an antiderivative, which may be obtained by term-by-term antidifferentiation. These facts allow us to build new power series representations for analytic function from old ones, much as is done in calculus.

Also in analogy with calculus, an analytic function f has a *unique* power/Laurent series representation which converges at any given point. [Note however that f can have different Laurent series representations centered at a particular point z_0, but the series will converge on *different* rings that share no point in common.]

Residues: From the formula for the coefficients of a Laurent series for an analyic function f we find in particular that

$\int_C f(z)\, dz = (2\pi i)$(the coefficient of $(z-z_0)^{-1}$ in the Laurent series)

if f is analytic on and inside of C (oriented counterclockwise), except (possibly) at $z = z_0$. This can be a very useful tool for computing many contour integrals, if finding the coefficients of the Laurent series requires less effort than the computation of the integral directly. For example, $f(z) = e^{1/z}$ is analytic everywhere except at $z = 0$, and by substituting $1/z$ into the Taylor series for e^z (and relying on the fact that the resulting Laurent series for f is <u>the</u> Laurent series for f), we can read off the coefficient, 1, of z^{-1} in the series for $e^{1/z}$ to conclude that $\int_C e^{1/z}\, dz = (2\pi i)(1)$ for any s.c.c traveling counterclockwise around $z = 0$.

The coefficient of $(z-z_0)^{-1}$ plays an important enough role in complex integration that we give it a name: a singularity z_0 of f (= a point where f is not analytic) is called *isolated* if f is analytic on some deleted neighborhood about z_0. For any isolated singularity z_0 of f, the

residue of f at $z = z_0$, denoted $\text{Res}_{z=z_0} f(z)$, is the coefficient of $(z-z_0)^{-1}$ in the Laurent series expansion of f for the deleted neighborhood of z_0. It also equals $1/(2\pi i) \int_C f(z)\, dz$ for any (small) simple closed curve C traveling counterclockwise around z_0. Using an earlier integration result (replacing a curve C with curves c_i inside it, where f is analytic between them), we get the

Residue Theorem: If C is a simple closed curve (oriented counterclockwise), and f is a function which is analytic on and inside of C, except at a finite number $z_1, \ldots, z_n$ of isolated singularities of f, none of which lie on C, then

$$\int_C f(z)\, dz = (2\pi i) \sum_{i=1}^{n} \text{Res}_{z=z_i} f(z)$$

This means that if we have a way to compute residues (i.e, coefficients of $(z-z_0)^{-1}$ in the Laurent series) at isolated singularities, then we have a way to compute contour integrals.

Isolated singularities $z = z_0$ come in three basic flavors, based on the behavior of the coefficients $f(z) = \displaystyle\sum_{n=-\infty}^{\infty} c_n (z-z_0)^n$ for $n < 0$.

If $c_n = 0$ for all $n < 0$, then by defining $f(z_0) = c_0$, f becomes analyic <u>at</u> $z = z_0$. z_0 is then called a *removable singularity.*

If $c_{-m} \neq 0$ for some $m > 0$ but $c_n = 0$ for all $n < -m$, then z_0 is called a *pole of order* m for f. In this case (which is the one we most often encounter), $f(z) = (z-z_0)^{-m} g(z)$ for some function g which is analytic at $z = z_0$. The residue of f at $z = z_0$ is then equal to the coefficient of $(z-z_0)^{-1}$ in the Taylor series for $(z-z_0)^{-m} g(z)$, which is equal to the coefficient of $(z-z_0)^{m-1}$ in the Taylor series for $g(z)$, which is $\dfrac{g^{(m-1)}(z_0)}{(m-1)!}$. That deserves being said twice!

If $(z-z_0)^m f(z) = g(z)$ is analytic at z_0 for some $m > 0$, then $\text{Res}_{z=z_i0} f(z) = \dfrac{g^{(m-1)}(z_0)}{(m-1)!}$

The final kind of singularity has $c_n \neq 0$ for infninitely many negative values of n. Such a singularity is called an *essential singularity.* For example, $z = 0$ is an essential singularity for $f(z) = e^{1/z}$, for $f(z) = z^{9004} \sin(1/z)$, and other similarly constructed functions. Computing residues at essential singularities usually involves identifying how to build it from other anaytic functions whose Laurent series we know, and constructing its Laurent series from the known one. The main fact about essential singularities (which we will not much use) is:

The Great Picard Theorem: If z_0 is an essential singularity for f, then except possibly for one value c, for every $\epsilon > 0$ the equation $f(z) = c$ has infinitely many solutions on the deleted neighborhood $0 < |z-z_0| < \epsilon$.

One immediate consequence of this is that if f is an entire function that is not a polynomial, then, with possibly one exception, $f(z) = c$ has infinitely many solutions, since $f(1/z)$ has an essential singularity at $z = 0$.

The residue at ∞: $\int_C f(z)\ dz$ can be computed from the residues of the singularities lying inside of C. But if there are fewer singularities lying outside of C, we would rather use them! To do so, we need to define the residue at ∞ of f, as the *negative* of the sum of the residues of f, so that the sum of "all" residues are 0, But the residue at ∞ can be computed in an alternate way: thinking of it as the integral of f around a very large clockwise curve, the substitution $w = 1/z$ will turn this integral into the integral of $-\frac{1}{w^2}f\left(\frac{1}{w}\right)$ around a very small (counterclockwise) curve around 0, so

$\text{Res}_{z=\infty} f(z) = -\text{Res}_{z=0}\frac{1}{z^2}f\left(\frac{1}{z}\right)$

Using this, countour integrals can be computed using residues inside or outside of C, as we wish.

Zeros of analytic functions: One observation we can draw from our discussion of poles has nothing to do with poles. If f is a non-constant analytic function with $f(z_0) = 0$, then the Taylor series for f cannot have all coefficients 0, so there is a smallest m so that $f(z) = (z - z_0)^m g(z)$ with $g(z_0) \neq 0$ ($m =$ the index of the first non-zero coefficient.) But then g is analytic (we can extract its power seires from f's), so is continuous, so $g(z) \neq 0$ for all z close enough to z_0. But $(z - z_0)^m \neq 0$ for all $z \neq z_0$, so we find that *the zeros of (non-constant) analytic functions are isolated.* The same sort of reasoning shows that at a pole z_0 of f, $|f(z)| \to \infty$ as $z \to z_0$.

In particular, this result implies that if f and g are analytic at $z = z_0$, and if for a sequence z_n converging to z_0 we have $f(z_n) = g(z_n)$, then $f = g$ in a neighborhood of z_0. This is because $f(z) - g(z)$ has a zero at z_0 which is not isolated! Essentially, an analytic function is determined by its values on any convergent sequence.

Residues are useful in computing contour integrals. But coupled with some facts we have developed along the way, residues can be used to compute many integrals from real variable calculus that are very difficult or impossible to do in any other way. In most cases that we have studied, these are improper integrals

$\int_{-\infty}^{\infty} f(x)\ dx$ or $\int_0^{\infty} f(x)\ dx$, which we approach by using the contours

$C(t) =$ the line segment from $-R$ to R, followed by the semicircle Re^{it} from $R = R + 0i$ to $-R$. The integral over C can be computed as $2\pi i$ times the sum of the residues lying inside of C. But for many functions, for R large enough, the integral of f over the semicircle can be shown to be small; all that we need, for example, is to know that $|zf(z)| \to 0$ as $|z| \to \infty$ (which we can establish if the 'degree' of the denominator is at least 1 higher than the degree of the numerator). The integral over C is then approximately $\int_{-R}^{R} f(x)\ dx$, whose limit as $R \to \infty$ is $\int_{-\infty}^{\infty} f(x)\ dx$. (Technically, it is what is really called the *Cauchy Principal Value* $= PV$ of the integral, P.V.$\int_{-\infty}^{\infty} f(x)\ dx$, but in most cases we can show that it is equal to the improper integral.)

Some extra precautions need to be taken when a multiple-valued function, such as $\log(z)$ or $z^{1/n}$ is involved, to take into account/avoid the branch curve that we need to introduce (and avoid!) in order to work with the function. For example, computing $\int_0^\infty \frac{\sqrt{x}}{(x^2+1)^2}\,dx$ can be done by choosing the branch curve to be the negative imaginary axis and integrating $\dfrac{\sqrt{z}}{(z^2+1)^2}$ over the curve C described above, *except that* we need to hop 'over' the origin (since it lies on the branch curve). The integral over C (which can be computed from the residue of the one singularity $z = i$ lying inside of C) is then the sum of

$\displaystyle\int_{-R}^{-1/R} \frac{\sqrt{x}}{(x^2+1)^2}\,dx$, $\displaystyle\int_{1/R}^{R} \frac{\sqrt{x}}{(x^2+1)^2}\,dx$, and integrals over a very small semicircle of radius $1/R$ and a very large semicircle of radius R.

The first integral is an imaginary number, since from our choice of branch of the square root, $\sqrt{x} = i\sqrt{|x|}$. The two semicircle will give integrals that are small, the first since the curve is short (and the function has small modulus near 0), and the second because the function is getting small fast enough to compensate for the increasing length of the curve. So our residue computation ends up giving us the sum $i\displaystyle\int_0^\infty \frac{\sqrt{x}}{(x^2+1)^2}\,dx + \int_0^\infty \frac{\sqrt{x}}{(x^2+1)^2}\,dx$, equating real and imaginary parts yields our desired integral.

Integrating (rational) functions of $\sin x$, $\cos x$: Such definite integrals from 0 to 2π can be treated quite generally: using the identities

$\sin t = \dfrac{e^{it} - e^{-it}}{2i}$ and $\cos t = \dfrac{e^{it} + e^{-it}}{2}$,

the substitution $z = e^{it}$ (so $dt = \dfrac{dz}{iz}$) can turn these integrals into integrals of (usually rational) functions of z over the unit circle $C(t) = e^{it}$, $0 \le t \le 2\pi$, which we can (often) compute using residues.