Math 423/823 Exercise Set 4

Due Thursday, Feb. 17

13. [BC#2.18.11] Show that if $T(z) = \dfrac{az+b}{cz+d}$ (where $a, b, c, d \in \mathbb{C}$ and $ad - bc \neq 0$ then

(a) if $c = 0$ then $\lim\limits_{z\to\infty} T(z) = \infty$.

(b) if $c \neq 0$ then $\lim\limits_{z\to\infty} T(z) = \dfrac{a}{c}$ and $\lim\limits_{z\to -d/c} T(z) = \infty$.

14. [BC#2.20.9] Let f be the function $f(z) = \begin{cases} (\overline{z})^2/z & \text{if } z \neq 0 \\ 0 & \text{if } z = 0 \end{cases}$.

    Show that f is not differentiable at 0, even though the limit of the difference quotient exists (and both agree) when you let $\Delta z \to 0$ along the vertical and horizontal axes; show that if you approach 0 along the line $h = k$ (where $\Delta z = h + ik$) you find a different limit.

15. [BC#2.23.6] Revisit problem #14 from the viewpoint of the Cauchy-Riemann equations. That is, write $f(z) = f(x + iy) = u(x, y) + iv(x, y)$ (noting that we define $u(0, 0) = v(0, 0) = 0$). Show that u_x, u_y, v_x, and v_y all exist at $(0, 0)$ *and* that they satisfy the Cauchy-Riemann equations at $(0, 0)$.

    [N.B. This shows that the CR equations alone are not enough to guarantee differentiability at a point.]

16. Let $f(z) = z^3 + 1$ and $a = \dfrac{1+\sqrt{3}i}{2}$, $b = \dfrac{1-\sqrt{3}i}{2}$. Show that there is *no* value of w on the line segment $\{\dfrac{1+t\sqrt{3}i}{2} \ : \ -1 \leq t \leq 1\}$ where $f'(w) = \dfrac{f(b)-f(a)}{b-a}$.

    (Note: It is probably most efficient to determine <u>all</u> of the w for which $f'(w) =$ that specific value, and show that none of those points lie on the line segment!)

    [N.B. Consequently, the direct analogue of the Mean Value Theorem (central to most theoretical results in calculus!) does not hold in general for analytic functions.]