Math 423/823 Exercise Set 7

Due Thursday, Mar. 31

25. Show that 'integration by parts' works with analytic functions: for any curve $\gamma(t)$, $a \le t \le b$, if f, g, f' and g' are all analytic along γ, then we have
$$\int_\gamma f(z)g'(z)\ dz = [f(\gamma(b))g(\gamma(b)) - f(\gamma(a))g(\gamma(a))] - \int_\gamma f'(z)g(z)\ dz$$
[Hint: $F(z) = f(z)g(z)$ is the antiderivative of what (analytic) function?]
[Note: we will shortly be learning that the analyticity of f' and g' follow from that of f and g, so the requirements on the derivatives are not, in the end, really neccesary...]

26. (Via the fundamental theorem of ('complex') calculus,)
Compute $\int_\gamma ze^{iz}\ dz$, where γ is the (unit) circular arc running from $z = 1$ to $z = i$.
[Hint! Problem #25 will help...]

27. [BC#4.49.7] Show that if $\gamma(t)$, $a \le t \le b$ is a simple closed curve traversed counter-clockwise (so that the bounded region R it encloses is always on the left), then
$$\frac{1}{2i}\int_C \overline{z}\ dz = \text{the area of the region } R.$$
[Hint: this is a "standard" consequence of Green's Theorem (from multivariate calculus), in disguise. Write $C(t) = x(t) + iy(t)$, and compute what the integral should be...note that the <u>real</u> part is an integral whose antiderivative we can write down!]

28. Evaluate the following integrals:

(a): $\displaystyle\int_{\gamma_1} \frac{dz}{z^2+1}$, where $\gamma_1(t) = 1 + e^{2\pi ti}$, $0 \le t \le 1$

(a): $\displaystyle\int_{\gamma_2} \frac{dz}{z^2+1}$, where $\gamma_2(t) = i + e^{2\pi ti}$, $0 \le t \le 1$

[Note: one of these requires the Cauchy integral formula (unless you have gotten very ambitious and are working them directly from the definition!).]