Math 423/823 Exercise Set 8

Due Thursday, April 21

29. [BC#4.52.10] Suppose that $w = f(z)$ is an *entire* function, and there is a (real) $A > 0$ so that for every $z \in \mathbb{C}$ we have $|f(z)| \leq A|z|$. Show that there is a (complex) a so that $f(z) = az$ for all z.

    [Note: the text suggests using Cauchy's inequality to show that $f''(z_0) = 0$ for all z_0 (so we can argue that f is linear). I <u>think</u> you can also show that $f'(z)$ is bounded, and then beat the problem over the head with Liouville's Theorem...]

30. [BC#5.62.4] Find the Laurent series expansions centered at $z = 0$ for the function
$$f(z) = \frac{1}{z^2(1-z)}$$
valid for (a) $0 < |z| < 1$, and (b) $1 < |z| < \infty$.

31. [BC#5.62.8] (a) If a is real and $|a| < 1$, show how to derive the Laurent series expansion
$$\frac{a}{z-a} = \sum_{n=1}^{\infty} \frac{a^n}{z^n}$$
valid for $|a| < |z| < \infty$.

(b) Setting $z = e^{i\theta}$ in the equation from (a), set the real and imaginary parts of each side equal to one another to show that
$$\sum_{n=1}^{\infty} a^n \cos n\theta = \frac{a\cos\theta - a^2}{1 - 2a\cos\theta + a^2} \quad \text{and} \quad \sum_{n=1}^{\infty} a^n \sin n\theta = \frac{a\sin\theta}{1 - 2a\cos\theta + a^2}$$
for any real a with $|a| < 1$ and any real θ.

32. [BC#6.71.2(part)] Use the Residue Theorem to evaluate the integral
$$\int_C z^2 e^{\frac{1}{z}}\, dz \ ,$$
where $C(t) = 3e^{it}$, $0 \leq t \leq 2\pi$.