Math 423/823 Final Exam Practice Problems: Solutions

[Note: These solutions were generated without much effort at checking, so the actual numbers could be off, but the essential process should be correct. Use them mostly to judge whether you have followed the process correctly... And if you do think some of the answers are wrong, let your instructor know!]

1. For $z = x + yi$, does 1^z always equal 1 ?

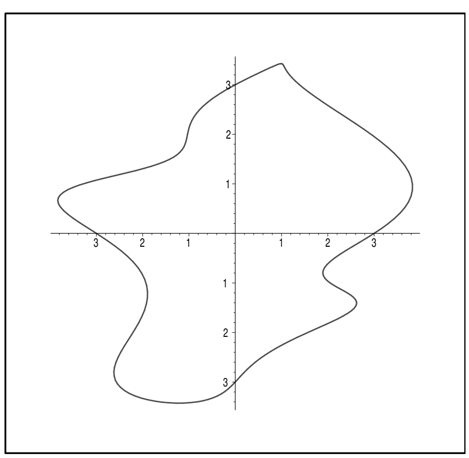
   By definition, $a^z = e^{z\log(a)}$. So $1^z = e^{z\log(1)}$. But $\log(1) = \ln|1| + i\arg(1) = 0 + i(0 + 2k\pi) = 2k\pi i$, depending on which branch of $\arg(z)$ we choose. So $1^z = e^{2k\pi iz}$ which, depending on our choice of k, need not equal 1. For $k = 0$, $1^z = e^{0z} = e^0 = 1$, but for, e.g., $k = 1$, $1^i = e^{2\pi ii} = e^{-2\pi} \neq 1$. So depending upon which value of $\log(1)$ that we choose, 1^z need not equal 1.

   Shorter, pithier solution: $1^{1/2}$ should be allowed to be -1, under any reasonable definition of exponentials, so, no.

2. Find the value of $\int_C f(z)\,dz$, where $f(z) = f(x + iy) = x^2 - iy^2$ and $C(t) = e^{it}$ for $0 \leq t \leq \pi$.

   Using the parametrization given, as $C(t) = e^{it} = \cos(t) + i\sin(t)$, we have $C'(t) = -\sin(t) + i\cos(t)$, so
   $\int_C f(z)\,dz = \int_0^\pi (\cos^2 t - i\sin^2 t)(-\sin t + i\cos t)\,dt = \int_0^\pi -\sin t\cos^2 t + i\cos^3 t + i\sin^3 t - i^2\cos t\sin^2 t\,dt = \int_0^\pi -\sin t\cos^2 t + \cos t\sin^2 t + i\cos t(1 - \sin^2 t) + i\sin t(1 - cos^2 t)\,dt = [(1/3)\cos^3 t + (1/3)\sin^3 t + i(t - (1/3)\sin^3 t) - i(t - (1/3)\cos^3 t)]|_0^\pi = [(-1/3) + 0 + i(\pi - 0) - i(\pi + (1/3))] - [(1/3) + 0 + i(0 - 0) - i(0 - (1/3))] = [-1/3 - (1/3)i] - [1/3 + (1/3)i] = -2/3 - (2/3)i$

3. Find the integral of the function $f(z) = \dfrac{z}{z^3 + 1}$ around the simple closed curve $C(t) = [3 + \sin(5t)]\cos t + i[3 + \sin(2t)]\sin t$, $0 \leq t \leq 2\pi$. [See figure below.]



$f(z)$ is analytic except where the denominator is zero, i.e., $z^3 = -1 = e^{i\pi}$,

so except for
$z = e^{i\pi/3} = z_1$, $z = e^{3i\pi/3} = e^{i\pi} = -1 = z_2$, and $z = e^{5i\pi/3} = e^{-i\pi/3} = z_3$.
In part, $z^3 + 1 = (z - z_1)(z - z_2)(z - z_3)$.

We note that each of these three points lie inside of the curve C; C travels around the origin at a distance of between $3 - 1 = 2$ and $3 + 1 = 4$, so encircles these three points, which all sit at a distance of 1 from the origin.

If we draw small circles C_i around each of the points z_i, in the counterclockwise direction, then from Cauchy's Theorem we know that

$$\int_C f(z)\ dz = \int_{C_1} f(z)\ dz + \int_{C_2} f(z)\ dz + \int_{C_3} f(z)\ dz\ .$$

But by the Cauchy Integral Formula,

$$\int_{C_1} f(z)\ dz = \int_{C_1} \frac{z}{(z - z_2)(z - z_3)} \frac{1}{z - z_1}\ dz = (2\pi i)\frac{z_1}{(z_1 - z_2)(z_1 - z_3)},$$

and by an identical argument

$$\int_{C_2} f(z)\ dz = (2\pi i)\frac{z_2}{(z_2 - z_1)(z_2 - z_3)} \text{ and } \int_{C_3} f(z)\ dz = (2\pi i)\frac{z_3}{(z_3 - z_1)(z_3 - z_2)}.$$

Putting these together, we get

$$\int_C f(z)\ dz = (2\pi i)[\frac{z_1}{(z_1 - z_2)(z_1 - z_3)} + \frac{z_2}{(z_2 - z_1)(z_2 - z_3)} + \frac{z_3}{(z_3 - z_1)(z_3 - z_2)}]$$

$$= (2\pi i)\frac{z_1(z_2 - z_3)}{(z_1 - z_2)(z_1 - z_3)(z_2 - z_3)} + \frac{-z_2(z_1 - z_3)}{(z_1 - z_12)(z_1 - z_3)(z_2 - z_3)} + \frac{z_3(z_1 - z_2)}{(z_1 - z_2)(z_1 - z_3)(z_2 - z_3)}]$$

$$= (2\pi i)[\frac{z_1(z_2 - z_3) - z_2(z_1 - z_3) + z_3(z_1 - z_2)}{(z_1 - z_2)(z_1 - z_3)(z_2 - z_3)}]\ .$$

But! $z_1(z_2 - z_3) - z_2(z_1 - z_3) + z_3(z_1 - z_2) = z_1z_2 - z_1z_3 - z_2z_1 + z_2z_3 + z_3z_1 - z_3z_2$
$= (z_1z_2 - z_2z_1) + (-z_1z_3 + z_3z_1) + (z_2z_3 - z_3z_2) = 0$, so

$$\int_C f(z)\ dz = 0\ (!).$$

[Alternative solutions include computing as a sum of residues, or by noting that for any of the roots z_i of z^3+1, $z^3+1 = (z-z_i)(z^2+z_iz+z_i^2)$ to simplify the residues/integrals, <u>or</u> computing the residue at ∞, instead.]

4. If $w = f(z)$ is analytic **and non-constant** on and inside of the simple closed curve C and, for some constant K, $|f(z)| = K$ for every point on C, show that there is a point z_0 inside of C where $f(z_0) = 0$.

   [Hint: Suppose not! Then show that we can apply the Maximum Principle to both $f(z)$ and $g(z) = \dfrac{1}{f(z)}$ and get ourselves into trouble!]

   Suppose that $f(z) \neq 0$ for every z on and inside of the curve C.

   Then $g(z) = \dfrac{1}{f(z)}$ is a quotient of analytic functions on and inside of C, and the denominator is never zero there, so $g(z)$ is analytic (and non-constant) on and inside of C.

   Also, on C, $|g(z)| = \dfrac{1}{K} = L$ for every z.

But then the Maximum Modulus Theorem (the maximum of the modulus must occur on the boundary of a domain), applied to f, tells us that $|f(z)| \leq K$ for every z on and inside of C (and the inequality is strict ($<$) inside of C), while applied to g, it tells us that $|g(z)| \leq L$ (with strict inequality inside of C).

But this second inequality, interpreted as a statement about f, says that $|f(z)| \geq K$ for every z on and inside of C. So $|f(z)|$ is <u>constantly</u> equal to K on and inside of C. But from a result in class, this implies that f is constant, contradicting one of our hypotheses. So it must be the case that $f(z) = 0$ somewhere on or inside of C.

[N.B: we could have also simply concluded that $|f(z)| < K$ and $|f(z)| > K$ inside of C, which is ridiculous! So one of our hypotheses must be false.]

5. Show that if $|z| = 1$, then for any complex number b we have $\left|\dfrac{z+b}{\overline{b}z+1}\right| = 1$.

We can show this several ways; one somewhat short way is to note that, since $|z| = 1$, we have

$|z+b| = |\overline{z+b}| = |\overline{z}+\overline{b}| = |\overline{z}+\overline{b}| \cdot |z| = |(\overline{z}+\overline{b})z| = |\overline{b}z+\overline{z}z| = |\overline{b}z+|z|^2| = |\overline{b}z+1|$,

so $\left|\dfrac{z+b}{\overline{b}z+1}\right| = \dfrac{|z+b|}{|\overline{b}z+1|} = \dfrac{|z+b|}{|z+b|} = 1$.

6. Find the values of $z = \sqrt{1+\sqrt{i}}$.

$\sqrt{i} = \sqrt{e^{\pi i/2}} = e^{\pi i/4}$ and $e^{5\pi i/4}$, which in rectangular coordinates are $\sqrt{2}/2 + \sqrt{2}/2i$ and $\sqrt{2}/2 - \sqrt{2}/2i$. So $1+\sqrt{i} = (2+\sqrt{2})/2 + \sqrt{2}/2i$ and $(2+\sqrt{2})/2 - \sqrt{2}/2i$. These both have modulus $(1/2)\sqrt{(2+\sqrt{2})^2 + (\sqrt{2})^2} = (1/2)\sqrt{8+4\sqrt{2}} = \sqrt{2+\sqrt{2}}$, and argument, well, some number α (and $-\alpha$). So the four values of $\sqrt{1+\sqrt{i}}$ are $\pm(2+\sqrt{2})^{1/4}e^{i\beta}$, where $\beta = \alpha/2$ and $-\alpha/2$.

7. Show that if f is an entire function and $f(x+2\pi) = f(x)$ for every <u>real</u> value of x, then $f(z+2\pi) = f(z)$ for every <u>complex</u> value z. [Hint: what can you say about $g(z) = f(z+2\pi) - f(z)$?]

The function $g(z) = f(z+2\pi) - f(z)$ has the property that $g(x) = f(x+2\pi) - f(x) = 0$ for every real number x. g is also an entire function, since $z \mapsto z+2\pi$ is analytic everywhere, and the composition andd diffference of analytic functions is analytic. But g then has zeros which are no isolated. This implies that g is constant near, say $z = 0$. But since we can use a little circle around 0 to compute the coeffficients of its Taylor series, which equals g everywhere, we have that g is zero <u>everywhere</u>. Therefore, $f(z) = f(z+2\pi)$ for every $z \in \mathbb{C}$.

8. Use residues to compute $\displaystyle\int_0^\infty \frac{dx}{x^6+1}$.

We can evaluate this integral as the limit, as $R \to \infty$ of $(1/2)\displaystyle\int_{-R}^{R} \frac{dx}{x^6+1}$ (since the integrand is an even function). This, in turn, can be thought of as 'half' of a contour integral which travels from $z = R$ to $z = -R$ by the (upper) semicircle S_R at radius

R. But $|\int_{S_R} \frac{1}{z^6+1}\ dz| = |\int_0^\pi \frac{1}{(Re^{it})^6+1} iRe^{it}\ dt| \leq \int_0^\pi |\frac{1}{(Re^{it})^6+1} iRe^{it}|\ dt \leq$ $|int_0^\pi \frac{R}{R^6-1}\ dt = \frac{R\pi}{R^6-1} \to 0$ as $R\to\infty$, so this will contribute nothing in the limit. So the integral we want is the limit of the contour integral along the x-axis and then along S_R. This is a closed surve, and so we can compute the contour integral using residues.

The singularities of $f(z) = \dfrac{1}{z^6+1}$ occur at the sixth roots of $-1 = e^{i\pi}$; there are six of them, and three of them lie inside of the closed curve we've built, at $z_1 = e^{i\pi/6}$, $z_2 = e^{i\pi/2} = i$, and $z_3 = e^{5i\pi/6}$. Each of these is a simple pole, and so their residues can be computed by factoring $z - z_i$ out of the denominator of f, and evaluating the rest at $z = z_i$. Doing this three times, for our three roots, and summing, will give us the value of our contour integral (which won't depend on R !), and so is equal to the improper integal we seek.

How we compute these residues is a bit ugly; but if we focus on computing their sum, we can streamline a bit. We have three roots z_1, z_2, z_3 as above and, with the other three roots, which happen to be $-z_1, -z_2, -z_3$, we want

$$\frac{1}{(z_1-z_2)(z_1-z_3)(z_1+z_1)(z_1+z_2)(z_1+z_3)} + \frac{1}{(z_2-z_1)(z_2-z_3)(z_2+z_1)(z_2+z+2)(z_2+z_3)} + \frac{1}{(z_3-z_1)(z_3-z_2)(z_3+z_1)(z_3+z+2)(z_3+z_3)}$$

$$= \frac{1}{(2z_1)(z_1^2-z_2^2)(z_1^2-z_3^2)} + \frac{1}{2z_2(z_2^2-z_1^2)(z_2^2-z_3^2)} + \frac{1}{2z_3(z_3^2-z_1^2)(z_3^2-z_2^2)}$$

$$= \frac{1}{2z_1z_2z_3(z_1^2-z_2^2)(z_1^2-z_3^2)(z_2^2-z_3^2)}(z_2z_3(z_2^2-z_3^2) - z_1z_3(z_1^2-z_3^2) + z_1z_2(z_1^2-z_2^2))$$

...and I am beginning to care less and less about the exact value... But let's forge ahead: $z_1^3 = e^{i\pi/2} = i$, $z_2^3 = i^3 = -i$, and $z_3^3 = e^{15i\pi/6} = i$. So $(z_2z_3(z_2^2-z_3^2) - z_1z_3(z_1^2-z_3^2) + z_1z_2(z_1^2-z_2^2)) = -iz_3 - iz_2 - iz_3 + iz_1 + iz_2 + iz_1 = 2i(z_1 - z_3)$. Nope, still don't care.

9. Use residues to compute $\displaystyle\int_0^\infty \frac{x^2dx}{x^4+1}$.

As in the previous problem, the integrand is an even function, so we really calculate $\displaystyle\int_0^\infty \frac{x^2dx}{x^4+1} = (1/2)\lim_{R\to\infty}\int_{-R}^R \frac{x^2dx}{x^4+1}$, which we again treat as 'half' of a contour integral, the other part being $\displaystyle\int_{S_R} \frac{z^2}{z^4+1}\ dz$.

As above, the integral $\displaystyle\int_{S_R} \frac{z^2}{z^4+1}\ dz = \int_0^\pi \frac{((Re^{it})^2}{(Re^{it})^4+1} iRe^{it}\ dt$ has modulus at most $\displaystyle\int_0^\pi \frac{R^2}{R^4-1} R\ dt = \frac{R^3\pi}{R^4-1}$, which goes to 0 as $R\to\infty$, so it will, again, play no role in the value, in the limit.

If we compute the resulting contour (along the x-axis, and then alson S_R) using residues, the function $f(z) = \frac{z^2}{z^4+1}$ has four singularities, two of which, $z_1 = e^{i\pi/4}$ and $z_3 = e^{3i\pi/4}$ lie inside of our chosen closed curve. [The other two roots are $\overline{z_1}$ and $\overline{z_3}$.] These are again simple poles, and so we can compute

$\displaystyle\int_{C_R} \frac{z^2}{z^4+1}\,dz = \int_{C_R} \frac{z^2}{(z^2+i)(z^2-i)}\,dz = \int_{C_1} \frac{z^2}{(z^2+i)(z^2-i)}\,dz + \int_{C_3} \frac{z^2}{(z^2+i)(z^2-i)}\,dz =$
$\displaystyle 2\pi i(\frac{z_1^2}{(z_1^2+i)(z_1-\overline{z_1})} + \frac{z_3^2}{(z_3^2-i)(z_3-\overline{z_3})}$.

But! $z_1 - \overline{z_1} = \sqrt{2}i = z_3 - \overline{z_3}$, $z_1^2 = i$ and $z_3^2 = -i$, and so

$\displaystyle\int_{C_R} \frac{z^2}{z^4+1}\,dz = 2\pi i(\frac{i}{(i+i)(\sqrt{2}i)} + \frac{-i}{(-i-i)(\sqrt{2}i)} = 2\pi i(2\frac{1}{2\sqrt{2}i}) = \pi\sqrt{2}.$

So! $\displaystyle\int_0^\infty \frac{x^2dx}{x^4+1} = (1/2)\lim_{R\to\infty}\int_{-R}^R \frac{x^2dx}{x^4+1} = (1/2)\lim_{R\to\infty}\pi\sqrt{2} = \frac{\pi}{\sqrt{2}}$.

10. Find the integral of $f(z) = \dfrac{z}{1+\overline{z}}$ over the line segment $\gamma(t) = t$, $0 \le t \le 1$.

From the parametrization we have $\gamma'(t) = 1$ and so $\displaystyle\int_C \frac{z}{1+\overline{z}}\,dz = \int_0^1 \frac{t}{1+\overline{t}}\,1\,dt =$
$\displaystyle\int_0^1 \frac{t}{1+t}\,dt = \int_0^1 1 - \frac{1}{1+t}\,dt - t - \ln(1+t)|_0^1 = 1 - \ln 2.$

[This problem would have been more 'interesting' if the curve had not been in the real line, so that complex conjugation would have actually changed the function...!]

11. Determine, for the branch of the analytic function $f(z) = z^{1/2}$ with domain all z except for $\{x + 0i : x \le 0$ and with $f(1) = 1$, whether or not $f(z_1z_2) = f(z_1)f(z_2)$ hold for every z_1, z_2 in the domain of f. Is there a different choice of branch cut which would change the answer?

The short answer is no, and no.

The point is that the argument of $z^{1/2}$ is half of the argument of z, which for the branch chosen, is taken between $-\pi$ and π. So two numbers z_i with argument close to, but less than, π will have $f(z_i)$ have argument close to $\pi/2$, and so their product will have argument close to pi. But z_1z_2 will have argument close to, but less than, 2π, which for f we will have to interpret, instead, as negative and small. So $f(z_1z_2)$ will have argument half that size, that is, small and negative. If you work this out with specific numbers, what you find is that $f(z_1)f(z_2) = -f(z_1z_2)$, and so is not equal to $f(z_1z_2)$. And we can make this problem happen with any branch of $z^{1/2}$ we might choose; you 'just' need to find two numbers on one side of the branch (i.e., on one side of the line determined by the branch ray) whose product is on the other side, and you can recreate the scenario above.

12. Write the function $f(z) = \dfrac{z}{z^2-4z+3}$ as a Laurent series which converges for $1 < |z| < 3$, and as (another!) Laurent series which converges for $3 < |z| < \infty$.

We have $f(z) = \dfrac{z}{(z-1)(z-3)}$, which, using partial fractions, is equal to $\dfrac{3}{2}\dfrac{1}{z-3} - \dfrac{1}{2}\dfrac{1}{z-1}$. We can write Laurent series for each piece, that works in each ring-shaped domain, and then add them together to get the needed series.

For $\frac{1}{z-3}$, we can write $\frac{1}{z-3} = -\frac{1}{3-z} = -\frac{1}{3}\frac{1}{1-(z/3)} = -\frac{1}{3}\sum_{n=0}^{\infty}(z/3)^n = \sum_{n=0}^{\infty} -\frac{1}{3^{n+1}}z^n$, which converges for $|z/3| < 1$, i.e., $|z| < 3$. For $|z| > 3$, that is, $|z/3| > 1$, that is, $|3/z| < 1$, we substitute $w = 3/z$ into $\frac{1}{1-w} = \sum_{n=0}^{\infty} w^n$ to get $\sum_{n=0}^{\infty}(3/z)^n = \sum_{n=0}^{\infty} 3^n z^{-n} = \frac{1}{1-(3/z)} = \frac{z}{z-3}$, so $\frac{1}{z-3} = \sum_{n=0}^{\infty} 3^n z^{-n-1}$, which is a Laurent series which converges for $|z| > 3$.

for $\frac{1}{z-1}$, we will only need the Laurent series which converges for $|z| > 1$, which we will use to obtain both series. This, again, requires substituting $w = 1/z$ into $\frac{1}{1-w} = \sum_{n=0}^{\infty} w^n$, to obtain $\sum_{n=0}^{\infty} z^{-n} = \frac{1}{1-(1/z)} = \frac{z}{z-1}$, so $\frac{1}{z-1} = \sum_{n=0}^{\infty} z^{-n-1}$, which converges for $|z| > 1$.

Therefore, for $1 < |z| < 3$, we have $f(z) = \frac{3}{2}\frac{1}{z-3} - \frac{1}{2}\frac{1}{z-1} = \frac{3}{2}\sum_{n=0}^{\infty} -\frac{1}{3^{n+1}}z^n - \frac{1}{2}\sum_{n=0}^{\infty} z^{-n-1}$,

while for $3 < |z|$, we have $f(z) = \frac{3}{2}\frac{1}{z-3} - \frac{1}{2}\frac{1}{z-1} = \frac{3}{2}\sum_{n=0}^{\infty} 3^n z^{-n-1} - \frac{1}{2}\sum_{n=0}^{\infty} z^{-n-1}$.

The diligent student can express each of these as a single sum...

13. Find the residue at $z = 1$ for the functions $f(z) = \frac{z}{z^2-1}$ and $g(z) = \frac{\sin(2\pi z)}{(z-1)^2}$.

These are probably mostly quickly done using the Cauchy Integral Formula. For $f(z) = \frac{z}{z^2-1} = \frac{\frac{z}{z+1}}{z-1}$, the residue is equal to $\frac{1}{2\pi i}\int_C \frac{\frac{z}{z+1}}{z-1}\,dz = \frac{z}{z+1}|_{z=1} = \frac{1}{2}$. [C is a small curve around $z = 1$. It would be instructive to find this by instead writing $f(z)$ as a Laurent series in $(z-1)$ ...]

For $g(z) = \frac{\sin(2\pi z)}{(z-1)^2}$, the residue is equal to $\frac{1}{2\pi i}\int_C \frac{\sin(2\pi z)}{(z-1)^2}\,dz = \frac{d}{dz}(\sin(2\pi z))|_{z=1} = 2\pi\cos(2\pi z)|_{z=1} = 2\pi\cos(2\pi) = 2\pi$.

14. Let C be any simple closed curve in the plane, oriented counterclockwise, and for z not on C, define

$$f(z) = \int_C \frac{s^3+2s}{(s-z)^3}\,ds\ .$$

Show that for every z inside of C, $f(z) = 6\pi i z$, while for every z outside of C, $f(z) = 0$.

If z is outside of C, then $g(s) = \frac{s^3+2s}{(s-z)^3}$ is analytic on an inside of C, since it is the quotient of analytic functions. Therefore, by the Cauchy-Goursat Theorem, $\int g(s)\,ds = 0$.

If z is inside of C, then setting $g(s) = s^3 + 2s$, we have

$f(z) = \int_C \frac{s^3+2s}{(s-z)^3}\,ds = \int_C \frac{g(s)}{s-z)^3}\,ds = 2\pi i\frac{g''(z)}{2!}$. Since $g''(s) = 6s$, we have $f(z) = 2\pi i(6z/2) = 6\pi iz$.

15. Show that if

$$f(z) = f(x+yi) = u(x,y) + iv(x,y) \text{ and } g(z) = g(x+yi) = p(x,y) + iq(x,y)$$

both satisfy the Cauchy-Riemann equations at $z = 0$, then $h(z) = f(z)g(z)$ <u>also</u> satisfies the CR-equations at $z = 0$.

[There is nothing at all special about 0; it was chosen for notational convenience.]

The real part of fg is $U = up - vq$, and the imaginary part of fg is $V = vp + uq$. So we wish to show that $U_x = V_y$ and $U_y = -V_x$. But since we know that $u_x = v_y$, $u_y = -v_x$, $p_x = q_y$, and $p_y = -q_x$, we find that

$U_x = (up-vq)_x = (up)_x - (vq)_x = (u_xp+up_x) - (v_xq+vq_x) = (v_yp+uq_y) - (-u_yq + v(-p_y)) = (uq_y + u_yp) + (v_yp + p_yv) = (uq)_y + (vp)_y = (uq+vp)_y = V_y$

...and the other CR equation is similar...

16. Show that setting $z = e^{it}$, we can rewrite $\dfrac{\cos 5t}{\cos t}$ as $z^4 - z^2 + 1 - z^{-2} + z^{-4}$.

Use this to find the value of $\displaystyle\int_0^{2\pi} \frac{\cos 5t}{\cos t}\,dt$ by converting to an integral over the unit circle $C(t) = e^{it}$, $0 \le t \le 2\pi$.

We know that $\cos t = (1/2)(e^{it}+e^{-it}) = (1/2)(z+z^{-1})$, and so $\cos(5t) = (1/2)(e^{i5t} + e^{-i5t}) = (1/2)(z^5 + z^{-5})$. Therefore,

$\dfrac{\cos(5t)}{\cos t} = \dfrac{(1/2)(z^5+z^{-5})}{(1/2)(z+z^{-1})} = \dfrac{z^5+z^{-5}}{z+z^{-1}} = \dfrac{z^{-5}}{z^{-1}}\dfrac{z^{10}+1}{z^2+1} = z^{-4}(z^8 - z^6 + z^4 - z^2 + 1) = z^4 - z^2 + 1 - z^{-2} + z^{-4}$

by polynomial long division. From this, we can compute:

$(*) = \displaystyle\int_0^{2\pi} \frac{\cos(5t)}{\cos t}\,dt = \int_0^{2\pi} (e^{it})^4 - (e^{it})^2 + 1 - (e^{it})^{-2} + (e^{it})^{-4}\,dt = \frac{1}{i}\int_0^{2\pi} (e^{it})^3 - (e^{it})^1 + (e^{it})^{-1} - (e^{it})^{-3} + (e^{it})^{-5}\,(ie^{it})\,dt$

This is the integral that results from a contour integral, along the unit circle C, with parametrization $z = e^{it}$, for the function $f(z) = i(z^3 - z + z^{-1} - z^{-3} + z^{-5})$. So:

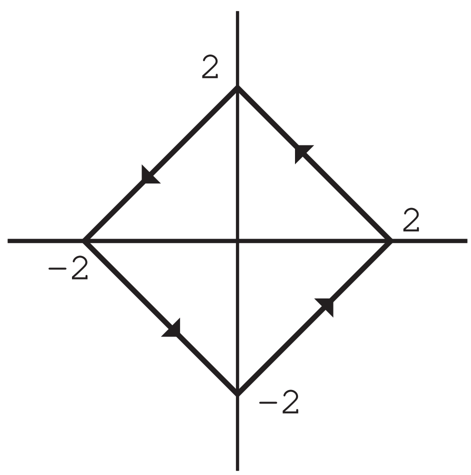
$(*) = \displaystyle\int_C i(z^3 - z + z^{-1} - z^{-3} + z^{-5})\,dz = 2\pi i(i) = -2\pi$, since by the Residue Theorem, the integral is equal to $2\pi i$ times the coefficient of z^{-1} in the Laurent series representation of the function, since the function is analytic on and inside of X except at $z = 0$.

17. Find the Laurent series expansion of the function $f(z) = \dfrac{z^3}{(z-1)^2}$ centered at $z = 0$, valid for $1 < |z| < \infty$.

$1 < |z|$ means $|1/z| < 1$, and so then using the geometric series, we have $\sum_{n=0}^{\infty}(1/z)^n = 1/(1-(1/z)) = 1/((z-1)/z) = z/(z-1)$ so $(z-1)^{-1} = 1/(z-1) = (1/z)\sum_{n=0}^{\infty}(1/z)^n = \sum_{n=\infty}^{-1} z^n$, valid for $|z| > 1$. Differentiating this term by term we have

$-(z-1)^{-2} = \sum_{n=\infty}^{-1} nz^{n-1}$, and so $f(z) = \dfrac{z^3}{(z-1)^2} = z^3 \sum_{n=\infty}^{-1} -nz^{n-1} = \sum_{n=\infty}^{-1} nz^{n+2} = \sum_{n=\infty}^{1} -(n-2)z^n = \sum_{n=\infty}^{1}(2-n)z^n$, which is, again, valid for $|z| > 1$.

18. Find the value of $\displaystyle\int_C \frac{dz}{(z^2+1)(2z+5)}$,
where C is the boundary of the 'diamond' $S = \{(x + iy : |x| + |y| \leq 2\}$, traversed counterclockwise (see figure below).



The singularities of the integrand occur at the roots of the denomenator, namely $z = i$, $z = -i$, and $z = -5/2$. Of these, the first two lie inside of C. So we can compute the integral as the sum of the integral of $f(z) = \dfrac{1}{(z^2+1)(2z+5)}$ around small circles surrounding i and $-i$, which, in turn, we can compute by the Cauchy Integral Formula. From this we get:

$$\int_C \frac{1}{(z^2+1)(2z+5)}\,dz = (2\pi i)\left(\frac{1}{(z+i)(2z+5)}\Big|_{z=i} + \frac{1}{(z-i)(2z+5)}\Big|_{z=-i}\right) = 2\pi i\left(\frac{1}{(2i)(2i+5)} + \frac{1}{(-2i)(-2i+5)}\right) = \pi\left(\frac{1}{2i+5} + \frac{1}{2i-5}\right) = \frac{4\pi i}{-2^2-5^2} = \frac{-4\pi i}{29} .$$

An alternate approach would be to use the singularities outside of C, namely $-5/2$ and ∞. The residue at ∞, because the denominator of the function is cubic, will be 0;

$\dfrac{1}{z^2}f(\dfrac{1}{z}) = \dfrac{1}{z^2(\frac{1}{z^2}+1)(\frac{2}{z}+5} = \dfrac{z}{(1+z^2)(2+5z)}$ has a removable singularity at $z = 0$.

So the integral we want is the negative of the residue of f at $z = -5/2$, that is,
$$-\int_C \frac{\frac{1}{(z^2+1)(2)}}{z+(5/2)}\,dz = -2\pi i \frac{1}{((-5/2)^2+1)(2)} = -2\pi i \frac{2}{29} = \frac{-4\pi i}{29} \ .$$

Some potentially useful formulas

$$\sin(z) = \frac{1}{2i}(e^{iz} - e^{-iz})$$

$$\cos(z) = \frac{1}{2}(e^{iz} + e^{-iz})$$

$$\arcsin(z) = -i\log(iz + \sqrt{1-z^2})$$

$$\arctan z = \frac{i}{2}\log\left(\frac{i-z}{i+z}\right)$$

$$\frac{1}{1-z} = \sum_{n=0}^{\infty} z^n \ , \text{ for } |z| < 1$$

$$\frac{1}{(1-z)^2} = \frac{d}{dz}\left(\frac{1}{1-z}\right)$$

$$\frac{d}{dz}\Big(\log(1-z)\Big) = \frac{-1}{1-z}$$

$$\sin z = \sum_{n=0}^{\infty} \frac{(-1)^n}{(2n+1)!} z^n$$

$$\cos z = \sum_{n=0}^{\infty} \frac{(-1)^n}{(2n)!} z^n$$

$$\sinh z = \sum_{n=0}^{\infty} \frac{1}{(2n+1)!} z^n$$

$$\cosh z = \sum_{n=0}^{\infty} \frac{1}{(2n)!} z^n$$

$$\frac{1}{z^2+1} = \sum_{n=0}^{\infty} (-1)^n z^{2n} \ , \text{ for } |z| < 1$$