Math 423/823 Topics Covered since the midterm exam

Don't forget the topics from the midterm exam!

Integration: Modelled on line integrals (integrate vector field around a parametrized curve), except instead of dot product we will use complex multiplication.

Complex-valued function of a real variable $\gamma(t) = x(t) + iy(t)$; think of (sometimes!) as a parametrized curve running around in the complex plane.

Derivative: $\gamma'(t) = x'(t) + iy'(t)$

Usual differentiation rules work: sum, difference, **product**, quotient

Chain Rule: $f(x+iy) = u + iv$, $\gamma(t) = x(t) + iy(t)$, then $[f(\gamma(t))]' = f'(\gamma(t)) \cdot \gamma'(t)$.

Integral (of a complex-valued function of a real variable): $f(t) = x(t) + iy(t)$

$\int_a^b f(t)\ dt = \int_a^b x(t)\ dt + i\int_a^b y(t)\ dt$

Usual integration rules work: constant multiple, sum, difference, u-substitution, integration by parts

Contours = parametrized curves in the complex plane: $z(t) = x(t) + iy(t)$

continuous: both $x(t)$ and $y(t)$ are continuous
differentiable: both $x(t)$ and $y(t)$ are differentiable

Typically: insist that contour is continuous, and differentiable except possibly at a finite number of points ("piecewise differentiable")

Simple path = path that never visits the same point in the plane twice (except that maybe the two endpoints agree)

Reparametrization: don't change where the path goes, just when! $f(\phi(t))$, where ϕ is either always increasing or always decreasing.

By the chain rule, $[f(\phi(t))]' = f\prime(\phi(t)) \cdot \phi'(t)$

Arclength: $|\gamma'(t)|$ = size of rate of change = 'speed'; $\int_a^b |\gamma'(t)|\ dt$ = length of the curve γ

Arclength is unchanged under reparametrization (why? u-subs)

Closed curve = parametrized curve $\gamma : [a,b] \to \mathbb{C}$ with $\gamma(a) = \gamma(b)$

Jordan Curve Theorem: Every *simple* closed curve γ separates the complex plane $\mathbb{C}$ into two regions R and S, exactly one of which is *bounded* (= for some R, every point lies within R of the origin). γ is called *positively oriented* if as we traverse γ (with increasing t) its bounded region always lies to our left. (This is also the 'counterclockwise' orientation around γ.)

Contour Integrals: For a complex-valued function $w = f(z)$ and a contour $z = \gamma(t)$, $a \leq t \leq b$, the contaour integral of f along γ is defined as

$$\int_\gamma f(z)\ dz = \int_a^b f(\gamma(t)\gamma'(t)\ dt$$

(if the limit exists). This integral was designed so that the number is unchanged under (orientation-preserving, i.e., $\phi'(t) > 0$) reparametrization of γ. [It picks up a minus sign (-) under orientation-reversing reparametrization.]

This integral satisfies the usual properties: behaves as expected under constant multiple, sums, differences; if the contour is split into pieces, the integral is the sum of the integrals over the pieces.

Motivating example: for $\gamma(t) = e^{it}$, $0 \leq t \leq 2\pi$, $\displaystyle\int_\gamma \frac{dz}{z} = 2\pi i$

The idea: $f(z) = 1/z$ has an antiderivative $F(z) = \text{Log}(z)$, except that this function is not defined on the negative x-axis. The failure of the antiderivative to be well-defined on all of the curve contributes to the non-zero result.

Fundamental Theorem of Complex Calculus: If $w = F(z)$ is analytic on a domain D, $F'(z) = f(z)$ on D, and $\gamma : [a, b] \to D$ is a contour that lies entirely in D, then

$$\int_\gamma f(z)\ dz = F(\gamma(b)) - F(\gamma(a))$$

In particular, the value of the integral depends only on the endpoints, and not on the particular curve used to join those endpoints. Note that this is really the same as saying that $\int_\gamma f(z)\ dz = 0$ for γ any closed path.

A basic inequality: $\displaystyle\left|\int_a^b z(t)\ dt\right| \leq \int_a^b |z(t)| dt$

So: If $|f(z)| \leq M$ along a curve γ of length L, then $\displaystyle\left|\int_\gamma f(z)\ dz\right| \leq LM$

Application: if $w = f(z)$ satisfies $|f(re^{i\theta}| \leq M(r)$ for all θ, for some function $M(r)$, and $rM(r) \to 0$ as $r \to \infty$, then for the curves $\gamma_r(t) = re^{it}$, $\theta_0 \leq t \leq \theta_1$, $\displaystyle\int_{\gamma_r} f(z)\ dz \to 0$ as $r \to \infty$.

The 'second' FTC, i.e., its converse!: If D is an open (connected) domain and f is a function so that for any curve $\gamma : [a, b] \to D$ in D, $\int_\gamma f(z)\ dz$ depends only on $\gamma(b)$ and $\gamma(a)$, then f has an antiderivative F. It can be given (choosing a 'base' point $z_0 \in D$) as:

$$F(z) = \int_\gamma f(w)\ dw \text{ where } \gamma \text{ is any curve from } z_0 \text{ to } z$$

So having an antiderivative on D is the same as having integral 0 around any closed curve in D.

Cauchy-Goursat Theorem: If $w = f(z)$ is analytic on and inside of the simple closed curve γ, then $\int_\gamma f(z)\ dz = 0$.

[Cauchy assumed $f'(z)$ is continuous, in order to use Green's Theorem!] This implies that if you are analytic, then you have an antiderivative.

Analyticity on and inside of the curve γ can be obtained 'for free' if f is analytic on a *simply-connected domain* D and γ lies entirely in D. D is *simply connected* if it is connected (any two points in D can be joined by a path in D) and for every simple closed curve [notation:

scc] γ in D the bounded region guaranteed by the Jordan curve theorem lies in D. A domain which is not simply-connected is (unfortunately) called *multiply connected*.

Example: the complex plane $\mathbb{C}$ is simply-connected, as are the upper half-plane $\{z = x+iy : y > 0\}$ and the points lying off of any one ray $\{re^{ia} : r \geq 0\}$ (for a fixed $a \in \mathbb{R}$). So, e.g., for any entire function f (think: $z^2, e^z, \sin z$, etc.) and any simple closed curve γ, $\int_\gamma f(z)\ dz = 0$.

If f is analytic on γ but not at every point inside of γ (which we assume is positively oriented), then is we surround the points of non-analyticity inside of γ with positively-oriented (and disjoint) simple loops $\gamma_1, \dots, \gamma_n$, so that f is analytic at every point that is inside of γ <u>and</u> outside of every one of the γ_i, then

$$\int_\gamma f(z)\ dz = \int_{\gamma_1} f(z)\ dz + \cdots + \int_{\gamma_n} f(z)\ dz$$

(This followed from Cauchy-Goursat by stitching together all of these loops into a single 'almost' simple loop.)

So, e.g., if δ lies inside of γ (i.e., lies in the region it bounds), and f is analyic at every point between the two scc's [and both are positively or negatively oriented], then $\displaystyle\int_\gamma f(z)\ dz = \int_\delta f(z)\ dz$. We often use this to trade an 'ugly' loop for something more standard (usually $\delta(t) = z_0 + r_0 e^{it}$ for fixed z_0 and r_0).

Cauchy Integral Formula: If f is analytic on and inside of a positively oriented scc γ, then

$$\frac{1}{2\pi i}\int_\gamma \frac{f(z)}{z - z_0}\ dz = f(z_0)$$

This has far-reaching consequences. It says that knowing the values of f along γ (and knowing that it is analytic inside of γ) is enough to be able to compute the value of f at every point inside of γ. It also gives us the tools to compute a wide range of integrals that are otherwise out of reach. It <u>also</u> gives us the tool to show the results below!

If f is analytic on and inside of (pos. or'd) γ, then $\displaystyle\frac{n!}{2\pi i}\int_\gamma \frac{f(z)}{(z - z_0)^{n+1}}\ dz = f^{(n)}(z_0)$.

In part, f is (infinitely) differentiable! so $f'(z)$ is analytic! But since $f = u + iv$ has $f'(z) = u_x + iv_x = v_y - iu_y$, this implies that u_x, v_x, u_y, v_y are all differentiable (and continuous). [These are facts we used before; now we have justified them.] Also, if you have an antiderivative, then you are analytic. So analytic functions are <u>precisely</u> the functions that have antiderivatives!

The same demonstration used for CIF shows that if γ is a simple closed curve and g is a function defined <u>on</u> γ, then for D the region lying inside of γ and the function $f : D \to \mathbb{C}$ defined by

$$f(z) = \frac{1}{2\pi i}\int_\gamma \frac{g(s)}{s - z}\ dz$$

is an analytic function on D ! It's derivative is $f'(z) = \frac{1}{2\pi i}\int_\gamma \frac{g(s)}{(s-z)^2}\ dz$ ...

Cauchy's Inequality: If F is analytic on and inside of a circle $C_R(t) = z_0 + Re^{it}$, and M_R is the maximum of F on the circle C_R, then $|f^{(n)}(z_0)| \le \frac{n!M_R}{R^n}$.

Liouville's Theorem: If f is an entire function and for some M we have $|f(z)| \le M$ for all z, then f is constant.

Fundamental Theorem of Algebra: Every polynomial with complex coefficients, of degree greater than 0, has a complex root.

Why? Otherwise $g(z) = [f(z)]^{-1}$ is a non-constant bounded entire function! The FTA implies that every polynomial completely factors as a product of linear polynomials. Over the reals, this means that every polynomial with real coefficients can be written as a product of linear and irreducible quadratic factors.

Some definite integral computations: Using $\sin t = \frac{e^{it} - e^{-it}}{2i}$ and $\cos t = \frac{e^{it} + e^{-it}}{2}$, we can write integrals $\int_0^{2\pi} F(\sin t, \cos t)\ dt$ as contour integrals around the unit circle $\gamma(t) = e^{it}$. Essentially, using the $z = e^{it}$, $dz = ie^{it}\ dt$, so $dt = \frac{dz}{iz}$, and $\sin t = \frac{z - z^{-1}}{2i}$ and $\cos t = \frac{z + z^{-1}}{2}$. So

$$\int_0^{2\pi} F(\sin t, \cos t)\ dt = \int_\gamma F\big(\frac{z - z^{-1}}{2i}, \frac{z + z^{-1}}{2}\big)\ \frac{dz}{iz}$$

By expressing the integrand as a rational function in z and finding the roots of the denominator which lie inside of the curve γ, we can (often) apply the Cauchy Integral Formula to evaluate the integral. For example, we can show that

$$\int_0^{2\pi} \frac{dt}{a + \sin t} = \frac{2\pi}{\sqrt{a^2 - 1}} \text{ for any } a > 1$$

Maximum Modulus Principle: If D is an open domain and f is a non-constant analytic function on D, then $|f(z)|$ never achieves a maximum value at any point in D.

As this is usually uysed, it is interpreted to as that if f is analytic on and inside of a scc γ (or between a collection of curves γ_i), then $|f(z)|$ (which is continuous!) must achieve is maximum <u>on</u> underbarthe underbarcurve γ (or on one of the curves γ_i).

As an application, for any function $u(x, y)$ that is harmonic on a domain D, building its harmonic conjugate v, and the analytic function $f(x + iy) = e^{u(x,y)+iv(x,y)}$, the MMP implies that u must achieve <u>its</u> maximum only on a boundary curve of D (or not at all...).

Power Series: Our basic guiding principle is that everything behaves the way it does for calculus/real variables, only more so!

$\sum_{n=0}^{\infty} a_n$ converges to L if the partial sums $S_N = \sum_{n=0}^{N} a_n$ are eventually as close to L as we would ever need them to be: $|S_N - L| \to 0$ as $N \to \infty$.

$\sum_{n=0}^{\infty}(a_n+b_n) = \sum_{n=0}^{\infty} a_n + \sum_{n=0}^{\infty} b_n$ hskip.2in $\sum_{n=0}^{\infty} ca_n = c\sum_{n=0}^{\infty} a_n$

Power Series: $f(z) = \sum_{n=0}^{\infty} a_n z^n$; domain = the z for which it converges!

Our basic workhorse series: $\sum_{n=0}^{\infty} z^n = \frac{1}{1-z}$; converges for $|z| < 1$, diverges for $|z| > 1$.

The big theorem: If $w = f(z)$ is analytic at $z = z_0$ (so there is a largest R so that f is analytic for all z with $|z - z_0| < R$ [possibly $R = \infty$], then the power series

$$\sum_{n=0}^{\infty} \frac{f^{(n)}(z_0)}{n!}(z-z_0)^n$$

converges for all $|z - z_0| < R$ and equals $f(z)$ there.

So, e.g., $f(z) = e^z$ is an entire function, and $f^{(n)}(0) = 1$ for all n, so e^z equals the power series you remember from calculus, for all complex numbers z. The same is true for the familiar power series representations of $\sin z, \cos z$, etc.

Prop: If $f(z) = \sum_{n=0}^{\infty} a_n(z-z_0)^n$ converges for some $z = z_1$, then setting $R = |z_1 - z_0|$, f converges for all $|z - z_0| < R$.

So if $f(z) = \sum_{n=0}^{\infty} a_n(z-z_0)^n$ does not converge for some $z = z_2$ (and $r = |z_2 - z_0|$), then f does not converge for any z with $|z - z_0| > r$, because if it did converge, it would have to converge at $z = z_2$, too. The closest point z_1 to z_0 for which f does not converge defines the *circle of convergence* $|z - z_0| = R = |z_1 - z_0|$; inside of this circle, the series converges; outside of the circle, it diverges.

Laurent series: Even when a function f fails to be analytic at a point $z = z_0$, if it is analytic in a *deleted* neighborhood $0 < |z - z_0| < R$ around z_0 we can still represent it as a series centered at z_0. We 'just' need to allow for negative exponents! A power series with both negative and (possibly) positive exponents is called a *Laurent series.*

Again, our basic workhorse for doing this is $f(z) = \frac{1}{1-z}$; for $|z| > 1$,

$$\frac{1}{1-z} = -\frac{1/z}{1-1/z} = -\frac{1}{z}\sum_{n=0}^{\infty}\left(\frac{1}{z}\right)^n = \sum_{n=-1}^{\infty}(-1)z^{-n} = \sum_{n=-\infty}^{-1}(-1)z^n,$$

since $|1/z| < 1$. Based on this, we can show:

If $w = f(z)$ is analytic on a ring $R_1 < |z - z_0| < r_2$ centered at $z = z_0$, then f can be expressed as a Laurent series

$$f(z) = \sum_{n=0}^{\infty} a_n(z-z_0)^n + \sum_{n=1}^{\infty} b_n(z-z_0)^{-n} = \sum_{n=-\infty}^{\infty} c_n(z-z_0)^n,$$

where, for C = any simple closed curve (oriented counterclockwise) lying in the ring and containing z_0 in its bounded complementary region, we have

$$c_n = \frac{1}{2\pi i} \int_C \frac{f(z)}{(z-z_0)^{n+1}}\, dz$$

Note that if f is actually analytic on all of $|z-z_0| < R_2$, then for $n \leq -1$, $\frac{f(z)}{(z-z_0)^{n+1}} = f(z)(z-z_0)^{-(n+1)}$ is also analytic (the exponent is non-negative), so $c_n = 0$ by the Cauchy-Goursat Theorem. So all of the coefficients of negative powers are zero.

Just as in calculus, inside of its circle(s) of convergence a power series $f(z) = \sum_{n=-\infty}^{\infty} c_n(z-z_0)^n$ is continuous, is differentiable, and its derivative can be obtained by term-by-term differentiation of its Taylor/Laurent series. This implies that a power series is analytic inside of its circle(s) of convergence. It then has an antiderivative, which may be obtained by term-by-term antidifferentiation. These facts allow us to build new power series representations for analytic function from old ones, much as is done in calculus.

Also in analogy with calculus, an analytic function f has a *unique* power/Laurent series representation which converges at any given point. [Note however that f can have different Laurent series representations centered at a particular point z_0, but the series will converge on *different* rings that share no point in common.]

Residues: From the formula for the coefficients of a Laurent series for an analyic function f we find in particular that

$\int_C f(z)\, dz = (2\pi i)$(the coefficient of $(z-z_0)^{-1}$ in the Laurent series)

if f is analytic on and inside of C (oriented counterclockwise), except (possibly) at $z = z_0$. This can be a very useful tool for computing many contour integrals, if finding the coefficients of the Laurent series requires less effort than the computation of the integral directly. For example, $f(z) = e^{1/z}$ is analytic everywhere except at $z = 0$, and by substituting $1/z$ into the Taylor series for e^z (and relying on the fact that the resulting Laurent series for f is the Laurent series for f), we can read off the coefficient, 1, of z^{-1} in the series for $e^{1/z}$ to conclude that $\int_C e^{1/z}\, dz = (2\pi i)(1)$ for any s.c.c traveling counterclockwise around $z = 0$. The coefficient of $(z-z_0)^{-1}$ plays an important enough role in complex integration that we give it a name: a singularity z_0 of f (= a point where f is not analytic) is called *isolated* if f is analytic on some deleted neighborhood about z_0. For any isolated singularity z_0 of f, the **residue** of f at $z = z_0$, denoted $\text{Res}_{z=z_0} f(z)$, is the coefficient of $(z-z_0)^{-1}$ in the Laurent series expansion of f for the deleted neighborhood of z_0. It also equals $1/(2\pi i) \int_C f(z)\, dz$ for any (small) simple closed curve C traveling counterclockwise around z_0. Using an earlier integration result (replacing a curve C with curves c_i inside it, where f is analytic between them), we get the

Residue Theorem: If C is a simple closed curve (oriented counterclockwise), and f is a function which is analytic on and inside of C, except at a finite number $z_1, \ldots, z_n$ of isolated singularities of f, none of which lie on C, then

$$\int_C f(z)\ dz = (2\pi i)\sum_{i=1}^{n} \mathrm{Res}_{z=z_i} f(z)$$

This means that if we have a way to compute residues (i.e, coefficients of $(z-z_0)^{-1}$ in the Laurent series) at isolated singularities, then we have a way to compute contour integrals.

Isolated singularities $z = z_0$ come in three basic flavors, based on the behavior of the coefficients $f(z) = \displaystyle\sum_{n=-\infty}^{\infty} c_n(z-z_0)^n$ for $n < 0$.

If $c_n = 0$ for all $n < 0$, then by defining $f(z_0) = c_0$, f becomes analytic <u>at</u> $z = z_0$. z_0 is then called a *removable singularity*.

If $c_{-m} \neq 0$ for some $m > 0$ but $c_n = 0$ for all $n < -m$, then z_0 is called a *pole of order m* for f. In this case (which is the one we most often encounter), $f(z) = (z-z_0)^{-m}g(z)$ for some function g which is analytic at $z = z_0$. The residue of f at $z = z_0$ is then equal to the coefficient of $(z-z_0)^{-1}$ in the Taylor series for $(z-z_0)^{-m}g(z)$, which is equal to the coefficient of $(z-z_0)^{m-1}$ in the Taylor series for $g(z)$, which is $\dfrac{g^{(m-1)}(z_0)}{(m-1)!}$. That deserves being said twice!

If $(z-z_0)^m f(z) = g(z)$ is analytic at z_0 for some $m > 0$, then $\mathrm{Res}_{z=z_i0} f(z) = \dfrac{g^{(m-1)}(z_0)}{(m-1)!}$

The final kind of singularity has $c_n \neq 0$ for infninitely many negative values of n. Such a singularity is called an *essential singularity.* For example, $z = 0$ is an essential singularity for $f(z) = e^{1/z}$, for $f(z) = z^{9004}\sin(1/z)$, and other similarly constructed functions. Computing residues at essential singularities usually involves identifying how to build it from other anaytic functions whose Laurent series we know, and constructing its Laurent series from the known one. The main fact about essential singularities (which we will not much use) is:

The Great Picard Theorem: If z_0 is an essential singularity for f, then except possibly for one value c, for every $\epsilon > 0$ the equation $f(z) = c$ has infinitely many solutions on the deleted neighborhood $0 < |z - z_0| < \epsilon$.

One immediate consequence of this is that if f is an entire function that is not a polynomial, then, with possibly one exception, $f(z) = c$ has infinitely many solutions, since $f(1/z)$ has an essential singularity at $z = 0$.

The residue at ∞: $\int_C f(z)\ dz$ can be computed from the residues of the singularities lying inside of C. But if there are fewer singularities lying outside of C, we would rather use them! To do so, we need to define the residue at ∞ of f, as the *negative* of the sum of the residues of f, so that the sum of "all" residues are 0, But the residue at ∞ can be computed in an alternate way: thinking of it as the integral of f around a very large <u>clockwise</u> curve, the substitution $w = 1/z$ will turn this integral into the integral of $-\dfrac{1}{w^2}f\left(\dfrac{1}{w}\right)$ around a very small (counterclockwise) curve around 0, so

$\text{Res}_{z=\infty} f(z) = -\text{Res}_{z=0} \frac{1}{z^2} f\left(\frac{1}{z}\right)$

Using this, countour integrals can be computed using residues inside or outside of C, as we wish.

Zeros of analytic functions: One observation we can draw from our discussion of poles has nothing to do with poles. If f is a non-constant analytic function with $f(z_0) = 0$, then the Taylor series for f cannot have all coefficients 0, so there is a smallest m so that $f(z) = (z - z_0)^m g(z)$ with $g(z_0) \neq 0$ ($m =$ the index of the first non-zero coefficient.) But then g is analytic (we can extract its power seires from f's), so is continuous, so $g(z) \neq 0$ for all z close enough to z_0. But $(z - z_0)^m \neq 0$ for all $z \neq z_0$, so we find that *the zeros of (non-constant) analytic functions are isolated.* The same sort of reasoning shows that at a pole z_0 of f, $|f(z)| \to \infty$ as $z \to z_0$.

In particular, this result implies that if f and g are analytic at $z = z_0$, and if for a sequence z_n converging to z_0 we have $f(z_n) = g(z_n)$, then $f = g$ in a neighborhood of z_0. This is because $f(z) - g(z)$ has a zero at z_0 which is not isolated! Essentially, an analytic function is determined by its values on any convergent sequence.

"Real" integral computations: Residues are useful in computing contour integrals. But coupled with some facts we have developed along the way, residues can be used to compute many integrals from real variable calculus that are very difficult or impossible to do in any other way. In most cases that we have studied, these are improper integrals $\int_{-\infty}^{\infty} f(x)\, dx$ or $\int_0^{\infty} f(x)\, dx$, which we approach by using the contours

$C(t) =$ the line segment from $-R$ to R, followed by the semicircle Re^{it} from $R = R + 0i$ to $-R$. The integral over C can be computed as $2\pi i$ times the sum of the residues lying inside of C. But for many functions, for R large enough, the integral of f over the semicircle can be shown to be small; all that we need, for example, is to know that $|zf(z)| \to 0$ as $|z| \to \infty$ (which we can establish if the 'degree' of the denominator is at least 1 higher than the degree of the numerator). The integral over C is then approximately $\int_{-R}^{R} f(x)\, dx$, whose limit as $R \to \infty$ is $\int_{-\infty}^{\infty} f(x)\, dx$. (Technically, it is what is really called the *Cauchy Principal Value* $= PV$ of the integral, P.V.$\int_{-\infty}^{\infty} f(x)\, dx$, but in most cases we can show that it is equal to the improper integral.) This works well with a rational function whose denomenator grows faster than its numerator, for example.

Some extra precautions need to be taken when a multiple-valued function, such as $\log(z)$ or $z^{1/n}$ is involved, to take into account/avoid the branch curve that we need to introduce (and avoid!) in order to work with the function. For example, computing $\int_0^{\infty} \frac{\sqrt{x}}{(x^2+1)^2}\, dx$ can be done by choosing the branch curve to be the negative imaginary axis and integrating $\frac{\sqrt{z}}{(z^2+1)^2}$ over the curve C described above, *except that* we need to hop 'over' the origin (since it lies on the branch curve). The integral over C (which can be computed from the residue of the one singularity $z = i$ lying inside of C) is then the sum of

$\displaystyle\int_{-R}^{-1/R} \frac{\sqrt{x}}{(x^2+1)^2}\,dx$, $\displaystyle\int_{1/R}^{R} \frac{\sqrt{x}}{(x^2+1)^2}\,dx$, and integrals over a very small semicircle of radius $1/R$ and a very large semicircle of radius R.

The first integral is an imaginary number, since from our choice of branch of the square root, $\sqrt{x} = i\sqrt{|x|}$. The two semicircle will give integrals that are small, the first since the curve is short (and the function has small modulus near 0), and the second because the function is getting small fast enough to compensate for the increasing length of the curve. So our residue computation ends up giving us the sum $i\displaystyle\int_0^\infty \frac{\sqrt{x}}{(x^2+1)^2}\,dx + \int_0^\infty \frac{\sqrt{x}}{(x^2+1)^2}\,dx$, equating real and imaginary parts yields our desired integral.

Integrating (rational) functions of $\sin x$**,** $\cos x$ **:** Such definite integrals from 0 to 2π can be treated quite generally: as before (with the Cauchy Integral Formula) using the identities

$\sin t = \dfrac{e^{it} - e^{-it}}{2i}$ and $\cos t = \dfrac{e^{it} + e^{-it}}{2}$,

the substitution $z = e^{it}$ (so $dt = \dfrac{dz}{iz}$) can turn these integrals into integrals of (usually rational) functions of z over the unit circle $C(t) = e^{it}$, $0 \leq t \leq 2\pi$, which we can (often) compute using residues.