Math 423/823 Exercise Set 4

Due Thursday, Feb. 15

14. Let $f(z) = z^3 + 1$ and $a = \dfrac{1+\sqrt{3}i}{2}$, $b = \dfrac{1-\sqrt{3}i}{2}$. Show that there is *no* value of w on the line segment $\{\dfrac{1+t\sqrt{3}i}{2} \ : \ -1 \le t \le 1\}$ where $f'(w) = \dfrac{f(b)-f(a)}{b-a}$.
(Note: It is probably most efficient to determine all of the w for which $f'(w) =$ that specific value, and show that none of those points lie on the line segment!)
[N.B. Consequently, the direct analogue of the Mean Value Theorem (central to most theoretical results in calculus!) does not hold in general for analytic functions.]

15. [BC#3.30.8, p.90] Find all of the values of $z \in \mathbb{C}$ so that
(a) $e^z = -2$ (b) $e^z = 1 + i$.

16. [BC#3.38.2 and 3, p.107, sort of] Show, from the definitions
$$\sin z = \frac{e^{iz} - e^{-iz}}{2i} \qquad \text{and} \qquad \cos z = \frac{e^{iz} + e^{-iz}}{2}$$
that (*) $\sin(z+w) = \sin z \cos w + \cos z \sin w$. Then, by treating w as a constant(!) and taking $\dfrac{d}{dz}$, show that $\cos(z+w) = \cos z \cos w - \sin z \sin w$.
[Hint: write out each side of the (first) equation (*) separately, and then work them towards one another!

17. [BC#3.38.7, p.108] Show that from the (alternate) formulas
$\sin z = \sin(x + yi) = \sin x \cosh y + i \cos x \sinh y$ and
$\cos z = \cos(x + yi) = \cos x \cosh y - i \sin x \sinh y$
that we can show that $|\sin z|^2 = \sin^2 x + \sinh^2 y$
and $|\cos z|^2 = \cos^2 x + \sinh^2 y$.
[The text suggests 'remembering' that $\sin^2 x + \cos^2 x = 1$
and $\cosh^2 y - \sinh^2 y = 1$ ...!]