Math 423/823 Exercise Set 5

Due Thursday, Feb. 22

18. [BC#2.27.2, p.79] Let the function $f(z) = u(x, y) + iv(x, y)$ be analytic in a domain D and consider the families of level curves

$$\mathcal{U} = \{\{(x, y) : u(x, y) = c_1\} : c_1 \in \mathbb{C}\} \text{ and } \mathcal{V} = \{\{(x, y) : v(x, y) = c_2\} : c_2 \in \mathbb{C}\}.$$

Show that wherever they meet, the curves in $\mathcal{U}$ are <u>orthogonal</u> to the curves in $\mathcal{V}$. That is, the slopes of the two curves, at a point of intersection, are negative reciprocals.

[Hint: for each level curve, treat it as implicitly defining y as a function of x and use the multivariate chain rule to, e.g., differentiate both sides of $u(x, y(x)) = c_1$ w.r.t. x. That is, show that if the graphs of $u(x, f(x)) = c_1$ and $v(x, g(x)) = c_2$ <u>meet</u>, then their (implicit) derivatives are negative reciprocals!]

19. [BC#3.30.12, p.90] For $z = x + yi$, write $\mathrm{Re}(e^{1/z})$ in terms of x and y. Explain why this function is harmonic in every domain D that does not contain 0.

20. [BC#3.33.5, p.96] Show that the set of values of $\log(i^{1/2})$ is $\left(n + \frac{1}{4}\right)\pi i$ for n any integer, and that the same is true for $\frac{1}{2}\log(i)$. Therefore,

$$\log(i^{1/2}) = \frac{1}{2}\log(i) ,$$

when these are treated as the <u>sets</u> of values of the multiple-valued function $\log(z)$.

[N.B.: Contrast this with the result of Example 5 on p.93 of the text, where we see that, in the same sense, $\log(i^2) \neq 2\log(i)$.]

21. Show that if $f(z) = f(x + yi) = u(x, y) + iv(x, y)$ is analytic, then the function

$$g(x, y) = u(x, y)v(x, y)$$

is **harmonic**. Show, also, that (one of) its harmonic conjugate(s) is

$$h(x, y) = \frac{1}{2}[(v(x, y))^2 - (u(x, y))^2].$$

[Hint: The product rule works for partial derivatives!]