Math 445

Take-home Exam (Exam 2)

Due on the instructor's desk by 3:00pm, Tuesday, November 26. You are not to discuss the exam, except on trivial matters, with anyone other than the instructor, until after you have turned in your solutions. The problems are given in approximately the order in which the material was presented in class; this is not necessarily a recommendation to work the problems in that order. Each needed computation should be carried out in full.

Show all work. How you get your answer is just as important, if not more important, than the answer itself. If you think it, write it!

1. (20 pts.) Show that for $x \notin \mathbb{Q}$ and h_n/k_n the convergents of its continued fraction expansion, that
$$k_n \left| xk_{n-1} - h_{n-1} \right| + k_{n-1} \left| xk_n - h_n \right| = 1$$

2. (15 pts.) Compute the continued fraction expansion of $\sqrt{39}$, and use this to find a solution to the Diophantine equation
$$x^2 - 39y^2 = 1$$
with $x, y \in \mathbb{Z}$ and $x \geq 1000$. For which of $N = 1, 2, 3, 4, 5$ does $x^2 - 39y^2 = N$ have a solution?

3. (15 pts.) Show that the Diophantine equation
$$x^2 - 39y^2 = 1776$$
has no solutions with $x, y \in \mathbb{Z}$

4. (15 pts.) Find a formula which gives all of the solutions (x, y) with $x, y \in \mathbb{Q}$ to the equation
$$x^2 + 3y^2 = 19$$
(i.e., find a pair of functions $x(r), y(r)$ so that $(x, y) \in \mathbb{Q}^2$ is a solution $\Leftrightarrow x = x(r), y = y(r)$ for some $r \in \mathbb{Q}$.)

5. (15 pts.) Show that if $n \equiv 7 \pmod 8$, then the equation
$$n = x^2 + y^2 + z^2$$
has no solutions with $x, y, z \in \mathbb{Z}$. Show that it is therefore not always possible to express the product of two sums of three squares (of integers) as a sum of three squares (of integers).

6. (20 pts.) Show that, for $n, m \in \mathbb{Z}$, if
$$x^2 + 2y^2 = m \text{ and } u^2 + 2v^2 = n$$
have solutions with $x, y, u, v \in \mathbb{Z}$, then the equation
$$z^2 + 2w^2 = nm$$
has a solution with $z, w \in \mathbb{Z}$.