Math 445 HW #5 Solutions

20. Continued fraction expansion of:

$$\frac{23}{38} = 0 + \frac{23}{38}$$

$$\frac{38}{23} = 1 + \frac{15}{23}$$

$$\frac{23}{15} = 1 + \frac{8}{15}$$

$$\frac{15}{8} = 1 + \frac{7}{8}$$

$$\frac{8}{7} = 1 + \frac{1}{7}$$

So $\frac{23}{38} = \langle 0, 1, 1, 1, 1, 7 \rangle$

$$\frac{112}{55} = 2 + \frac{2}{55}$$

$$\frac{55}{2} = 27 + \frac{1}{2}$$

So $\frac{23}{38} = \langle 2, 27, 2 \rangle$

21. If $(a,b)=1$ then the continued fraction expansion of $\frac{a}{b}$ has length at most b.

$\frac{a}{b} = \langle a_0, \ldots, a_n \rangle$ where the a_i are the quotients in the Euclidean algorithm for a,b:

$a = a_0 b + r_0$, $b = a_1 r_0 + r_1$, etc. So the length, n, of the continued fraction expansion of a/b is the number of quotients taken in the Euclidean algorithm. But since $b > r_0 > r_1 > \cdots > r_n \geq 0$ are all integers, there are at most b steps in the algorithm, so $n \leq b$.

22. Find the continued fraction expansion of $\sqrt{15}$.

$a_0 = \lfloor\sqrt{15}\rfloor = 3$, $x_0 = \sqrt{15} - 3$. $\frac{1}{\sqrt{15}-3} = \frac{\sqrt{15}+3}{15-9} = \frac{\sqrt{15}+3}{6}$,

so $a_1 = \left\lfloor \frac{\sqrt{15}+3}{6} \right\rfloor = 1$, $x_1 = \frac{\sqrt{15}+3}{6} - 1 = \frac{\sqrt{15}-3}{6}$.

$\frac{6}{\sqrt{15}-3} = \frac{6(\sqrt{15}+3)}{15-9} = \sqrt{15}+3$, so $a_2 = \lfloor\sqrt{15}+3\rfloor = 6$, $x_2 = \sqrt{15}-3 = x_0$.

So everything from this point on will repeat;

$a_3 = \left\lfloor \frac{1}{x_2} \right\rfloor = \left\lfloor \frac{1}{x_0} \right\rfloor = 1$, $x_3 = x_1$, $a_4 = \left\lfloor \frac{1}{x_3} \right\rfloor = \left\lfloor \frac{1}{x_1} \right\rfloor = 6$, etc.

So $\sqrt{15} = \langle 3,1,6,1,6,1,6,\ldots \rangle = \langle 3,\overline{1,6} \rangle$

Convergents: $\frac{3}{1}, \frac{4}{1}, \frac{27}{7}, \frac{31}{8}, \frac{213}{55}, \frac{244}{63}$

23. Find the continued fraction expansion of $\sqrt{23}$.

$a_0 = \lfloor\sqrt{23}\rfloor = 4$, $x_0 = \sqrt{23} - 4$. $\frac{1}{\sqrt{23}-4} = \frac{\sqrt{23}+4}{23-16} = \frac{\sqrt{23}+4}{7}$

$a_1 = \left\lfloor \frac{\sqrt{23}+4}{7} \right\rfloor = 1$, $x_1 = \frac{\sqrt{23}-3}{7}$. $\frac{7}{\sqrt{23}-3} = \frac{7(\sqrt{23}+3)}{23-9} = \frac{\sqrt{23}+3}{2}$

$a_2 = \left\lfloor \frac{\sqrt{23}+3}{2} \right\rfloor = 3$, $x_2 = \frac{\sqrt{23}-3}{2}$. $\frac{2}{\sqrt{23}-3} = \frac{2(\sqrt{23}+3)}{23-9} = \frac{\sqrt{23}+3}{7}$

$a_3 = \left\lfloor \frac{\sqrt{23}+3}{7} \right\rfloor = 1$, $x_3 = \frac{\sqrt{23}-4}{7}$. $\frac{7}{\sqrt{23}-4} = \frac{7(\sqrt{23}+4)}{23-16} = \sqrt{23}+4$

$a_4 = \lfloor\sqrt{23}+4\rfloor = 8$, $x_4 = \sqrt{23}-4 = x_0$. So everything will now repeat; $\sqrt{23} = \langle 4,1,3,1,8,1,3,1,8,\ldots \rangle = \langle 4,\overline{1,3,1,8} \rangle$

Convergents: $\frac{4}{1}, \frac{5}{1}, \frac{19}{4}, \frac{24}{5}, \frac{211}{44}, \frac{235}{49}$

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24. If $n \in \mathbb{N}$, $\sqrt{n} \notin \mathbb{Q}$, then for
$x = \langle a_0, a_1, \ldots, a_{k-1}, a_k + x_k \rangle$, $\quad x_k = \dfrac{\sqrt{n} - a}{b}$ where $b \mid n - a^2$. So the period of $\sqrt{n}$ is at most $n \lfloor \sqrt{n} \rfloor$.

By induction: $x_0 = \sqrt{n} - \lfloor \sqrt{n} \rfloor = \sqrt{n} - a_0 = \dfrac{\sqrt{n} - a_0}{1}$ with $1 \mid n - a_0^2$ ✓.

Suppose $x_k = \dfrac{\sqrt{n} - a}{b}$ with $b \mid n - a^2$. Then

$$\frac{1}{x_k} = \frac{b}{\sqrt{n} - a} = \frac{\sqrt{n} + a}{c} \quad \text{where } bc = (\sqrt{n} - a)(\sqrt{n} + a) = n - a^2.$$

Then $a_{k+1} = \left\lfloor \dfrac{1}{x_k} \right\rfloor$ and $x_{k+1} = \dfrac{\sqrt{n} + a}{c} - a_{k+1} = \dfrac{\sqrt{n} + (a - a_{k+1}c)}{c}$

So $x_{k+1} = \dfrac{\sqrt{n} - (a_{k+1}c - a)}{c}$ and

$$n - (a_{k+1}c - a)^2 = (n - a^2) + 2a a_{k+1} c - a_{k+1} c^2$$
$$= bc + 2a a_{k+1} c - a_{k+1} c^2 = c(b + 2a a_{k+1} - a_{k+1} c)$$

so $c \mid n - (a_{k+1}c - a)^2$, so x_{k+1} has the required form. wait: I can't just assume that

So by induction, $x_k = \dfrac{\sqrt{n} - a}{b}$ with $b \mid n - a^2$.

Then we have, in particular, ~~b~~ $1 \le b \le n$, and $a \in \mathbb{Z}$ and $1 \le a \le \sqrt{n}$, so $1 \le a \le \lfloor \sqrt{n} \rfloor$. So there are at most n choices for b, and $\lfloor \sqrt{n} \rfloor$ choices for a, so x_k takes on only $n \lfloor \sqrt{n} \rfloor$ different values. So two x_k's will repeat values, before we reach $k = n \lfloor \sqrt{n} \rfloor$. At that point, all values for the a_i's will begin to repeat, so the period (the time between repeats) is at most $n \lfloor \sqrt{n} \rfloor$. ///

Oops: My original solution was a little incomplete!
Thanks to Mark Stigge for pointing this out.

$$\sqrt{n} = \langle a_0, \ldots, a_{1\pm}x_n \rangle$$

(Induction) $x_n = \frac{\sqrt{n}-a}{b} > 0$ with $b \mid n-a^2$, and $a^2 < n$.

Claim: $a > 0$.

Pf: Induction! $\frac{1}{x_n} = \frac{b}{\sqrt{n}-a} = \frac{\sqrt{n}+a}{c}$, $bc = n-a^2$ with $0 < a < \sqrt{n}$.

$$\frac{\sqrt{n}+\sqrt{n}}{c} > \frac{\sqrt{n}+a}{c} > 1 \implies c < \sqrt{n}+a < 2\sqrt{n}.$$

$$\alpha = \left\lfloor \frac{\sqrt{n}+a}{c} \right\rfloor \qquad x_{n+1} = \frac{\sqrt{n}+a}{c} - \alpha = \frac{\sqrt{n}-(\alpha c - a)}{c}$$

Claim $\alpha c - a > 0$, ie.

$$a < \left\lfloor \frac{\sqrt{n}+a}{c} \right\rfloor c \quad \text{ie.} \quad \frac{a}{c} < \left\lfloor \frac{\sqrt{n}+a}{c} \right\rfloor$$

If $a < c$ then done, $\frac{a}{c} < 1 \le \left\lfloor \frac{\sqrt{n}+a}{c} \right\rfloor$

If $a \ge c$, then $c \le a < \sqrt{n}$, so

$$\left\lfloor \frac{\sqrt{n}+a}{c} \right\rfloor = \left\lfloor \frac{\sqrt{n}}{c} + \frac{a}{c} \right\rfloor \ge \left\lfloor 1 + \frac{a}{c} \right\rfloor = 1 + \left\lfloor \frac{a}{c} \right\rfloor > \frac{a}{c}$$

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