Math 445 H.W. #6 Solutions

25. Show that there is an irrational number x so that

$$|x - \tfrac{a}{b}| < |x - \tfrac{h_n}{k_n}| \quad \text{and} \quad b < k_{n+1} \text{ for suitable values of } a, b.$$

$x = \sqrt{2} = \langle 1, \overline{2} \rangle$ works: its first few convergents are
$\frac{0}{1}, \frac{1}{0}, \frac{1}{1}, \frac{3}{2}, \frac{7}{5}, \frac{17}{12}, \ldots$ and, since $\sqrt{2} = 1.415\ldots$

$|\sqrt{2} - \frac{3}{2}| = |1.5 - 1.415\ldots| > .084$, while

$|\sqrt{2} - \frac{4}{3}| = |1.415\ldots - 1.333\ldots| < .083$, so $|\sqrt{2} - \frac{4}{3}| < |\sqrt{2} - \frac{3}{2}|$

and $3 < 5$.

<u>OR</u>

$x = \sqrt{3} = \langle 1, \overline{1, 2} \rangle$ works: the first few convergents are
$\frac{0}{1}, \frac{1}{0}, \frac{1}{1}, \frac{2}{1}, \frac{5}{3}, \frac{7}{4}, \ldots$ and since $\sqrt{3} = 1.732\ldots$

$|\sqrt{3} - \frac{2}{1}| = |2 - 1.732\ldots| > .267$, while

$|\sqrt{3} - \frac{3}{2}| = |1.732 - 1.5| < .233$, so

$|\sqrt{3} - \frac{3}{2}| < |\sqrt{3} - \frac{2}{1}|$ and $2 < 3$.

It seems like nearly any (irrational) x would work?

26. Two solutions to $x^2-21y^2=1$

$4<\sqrt{21}<5$:

	$a_0=4$	$x_0=\sqrt{21}-4$
$\xi_1=\frac{\sqrt{21}+4}{5}$	$a_1=1$	$x_1=\frac{\sqrt{21}-1}{5}$
$\xi_2=\frac{\sqrt{21}+1}{4}$	$a_2=1$	$x_2=\frac{\sqrt{21}-3}{4}$
$\xi_3=\frac{\sqrt{21}+3}{3}$	$a_3=2$	$x_3=\frac{\sqrt{21}-3}{3}$
$\xi_4=\frac{\sqrt{21}+3}{4}$	$a_4=1$	$x_4=\frac{\sqrt{21}-1}{4}$
$\xi_5=\frac{\sqrt{21}+1}{5}$	$a_5=1$	$x_5=\frac{\sqrt{21}-4}{5}$
$\xi_6=\sqrt{21}+4$	$a_6=8$	$x_6=\frac{\sqrt{21}-4}{1}=x_0$

$\Rightarrow \frac{3524}{769},\ \frac{6049}{1320}$

(5), (1)

Convergents: $\frac{0}{1}, \frac{1}{0}, \frac{4}{1}, \frac{5}{1}, \frac{9}{2}, \frac{23}{5}, \frac{32}{7}, \frac{55}{12}, \frac{472}{103}, \frac{527}{115}, \frac{999}{218}, \frac{2525}{551},$

value of $h_m^2-21k_m^2$: (1) (-5) (4) (-3) (4) (-5) (1) (5) (4) (-3) (4)

So $55^2-21\cdot(12)^2=1$ & $\boxed{x=55,\ y=12}$ is a solution

From the above (computing convergents), $\boxed{x=6049,\ y=1320}$ is also a solution.

OR: $(55+\sqrt{21}\cdot 12)^2=(55^2+21\cdot 12^2)+\sqrt{21}(2\cdot 55\cdot 12)$

$=(3025+3024)+\sqrt{21}(110\cdot 12)$

$=6049+\sqrt{21}\cdot 1320$

So $x=6049,\ y=1320$ is a solution.

HGC

27. For which $1 \le N \le \sqrt{33}$ does $x^2 - 33y^2 = N$ have a solution?

$5 < \sqrt{33} < 6$:

$$a_0 = 5 \quad x_0 = \sqrt{33} - 5$$

$$\xi_1 = \frac{\sqrt{33}+5}{8} \quad a_1 = 1 \quad x_1 = \frac{\sqrt{33}-3}{8}$$

$$\xi_2 = \frac{\sqrt{33}+3}{3} \quad a_2 = 2 \quad x_2 = \frac{\sqrt{33}-3}{3}$$

$$\xi_3 = \frac{\sqrt{33}+3}{8} \quad a_3 = 1 \quad x_3 = \frac{\sqrt{33}-5}{8}$$

$$\xi_4 = \sqrt{33}+5 \quad a_4 = 10 \quad x_4 = \sqrt{33}-5 = x_0$$

So $\sqrt{33} = \langle 5, \overline{1,2,1,10} \rangle$. The values of $h_m^2 - 33k_m^2$ will be:

$m=0$	-8
$m=1$	3
$m=2$	-8
$m=3$	1

and then repeat.

So among $1,2,3,4,5 \le \sqrt{33}$, 1 and 3 will occur as values of $x^2 - 33y^2$; 4 will occur because it is a perfect square $(2^2 - 33(0)^2 = 4)$. Since 2 and 5 cannot occur as values of $h_m^2 - 33k_m^2$ for any m,

$x^2 - 33y^2 = N$ has no solutions for $N = 2, 5$

(also, for $N = -1, -2, -3, -4, -5$)

$x^2 - 33y^2 = N$ has solutions for $N = 1, 3, 4$.