Math 445 H.W. #7 Solutions

28. If $k \geq 2$, then $\exists\, x, y \geq 0$ with $2x+3y=k$.

By induction: $2 = 2\cdot 1 + 3\cdot 0$, $3 = 2\cdot 0 + 3\cdot 1$, $4 = 2\cdot 2 + 3\cdot 0$; if all $n<k$ can be expressed as $2x+3y=n$, $x,y \geq 0$, then $k-2 = 2x_0+3y_0$, so $k = 2(x_0+1)+3y_0$ can be expressed as desired (or $k-2<2 \Rightarrow k<4 \Rightarrow k=2$ or 3, which are dealt with above). So k can be expressed as $2x+3y=k$, $x,y\geq 0$. So by induction, we are done.

29. N is powerful if p prime $p|N \Rightarrow p^2|N$. Show N is powerful $\Longleftrightarrow N = a^2b^3$ for some $a,b \in \mathbb{Z}$.

Pf: ($\Leftarrow$) If $N=a^2b^3$ and p prime, $p|N=a^2b^3$ then either $p|a^2$ or $p|b^3$, so $p|a$ or $p|b$. But $p|a \Rightarrow p^2|a^2 \Rightarrow p^2|N$, and $p|b \Rightarrow p^3|b^3 \Rightarrow p^3|N$. So $p|N \Rightarrow p^2|N$.

($\Longrightarrow$) Suppose $N=p^\alpha$ for some α. Then N powerful $\Rightarrow$ $\alpha \geq 2 \Rightarrow \alpha = 2a+3b$ for some $a,b\geq 0$, so $N = p^\alpha = p^{2a+3b} = (p^a)^2(p^b)^3 = m^2n^3$ for some $m,n\in\mathbb{Z}$.

In general, $N = p_1^{k_1}\cdots p_r^{k_r}$ is powerful, then for each i, $(p_i|N \Rightarrow p_i^2|N) \Rightarrow k_i \geq 2$. So each exponent in the prime factorization of N is ≥ 2. But from the ~~~~ above, for each i $p_i^{k_i} = a_i^2 b_i^3$ for some $a_i, b_i \in \mathbb{Z}$,

so

$N = p_1^{k_1} \cdots p_r^{k_r} = (a_1^2 b_1^3) \cdots (a_r^2 b_r^3) = (a_1 \cdots a_r)^2 (b_1 \cdots b_r)^3$
$= A^2 B^3$ for $A, B \in \mathbb{Z}$, as desired. //

30. For which n does $x^2 - y^2 = n$ have a solution?

If $x = y$, then $x^2 - y^2 = y^2 - y^2 = 0$.
If $x = y+1$, then $x^2 - y^2 = (y+1)^2 - y^2 = 2y+1$; over all $y \in \mathbb{Z}$, this gives all odd numbers.
If $x = y+2$, then $x^2 - y^2 = (y+2)^2 - y^2 = 4y+4 = 4(y+1)$; over all $y \in \mathbb{Z}$, this gives all multiples of 4.

There only remains the odd multiples of 2, ~~i.e.~~ i.e. $2 \times (\text{odd } \#)$ i.e. $4n+2$ for $n \in \mathbb{Z}$. But $x^2 - y^2 = 4n+2$ has no solutions; since $x^2 - y^2$ would be even, we must have x, y both odd or both even; but then $x = 2r+1$, $y = 2s+1 \implies$ $x^2 - y^2 = (2r+1)^2 - (2s+1)^2 = 4(r^2 + r - s^2 - s)$ is a multiple of 4, and $x = 2r, y = 2s \implies x^2 - y^2 = (2r)^2 - (2s)^2 = 4(r^2 - s^2)$ is also a multiple of 4. So $2 \mid x^2 - y^2 \implies 4 \mid x^2 - y^2$, so no $n \equiv_4 2$ has a solution. So

$$x^2 - y^2 = n \text{ has a solution} \iff 2 \nmid n \text{ or } 4 \mid n. \quad //$$

31. For every n, $x^2 + z^2 = n + y^2$ has a solution.

Rewrite this as $x^2 - y^2 = n - z^2$; there exist x, y satisfying this $\iff$ there is a $z \in \mathbb{Z}$ so that either $2 \nmid n - z^2$ or $4 \mid n - z^2$.

But if n is <u>odd</u>, set $z=0$ so $n-z^2=n$ and $2\nmid n$.
If n is <u>even</u>, set $z=1$, so $n-z^2=n-1$ and $2\nmid n-1$.

So $x^2-y^2=n-z^2$ always has a solution (with $z=0$ or $z=1$, in fact...)

32. The equation $x^2+y^4=z^2$ has infinitely many solutions with $\gcd(x,y)=1$ and

(a) y odd | (b) y even.

Every solution of $x^2+(y^2)^2=z^2$ with y (hence y^2)

(a) odd must be of the form

$y^2=r^2-s^2$
$x=2rs$
$z=r^2+s^2$

(b) even

$x=r^2-s^2$
$y^2=2rs$
$z=r^2+s^2$

with r and s of opposite parity

(a) So we <u>need</u> r and s with $y^2=r^2-s^2$, ie.

$y^2+s^2=r^2$, y odd.

But setting

$y=u^2-v^2$ $s=2uv$, $r=u^2+v^2$

then defining x,z accordingly we get $x^2+y^4=z^2$, y odd, for infinitely many values of y (by varying r,s)

(b) So we need r and s with

$y^2=2rs$

Note that $y=2^d a$ with a odd for some d.

Then $y^2=2^{2d}a^2=2(2^{2d-1})a^2$

So setting $r=2^{2d-1}$ even and $s=a^2$ odd

and defining x,y, and z accordingly we get infinitely many (by varying a,d) solutions with y even.

As usual, starting the process at the bottom with relatively prime numbers will yield $(x,y)=1$.

On the left, $(u,v)=1 \implies (r,s)=1$ (a prime dividing u & v divides r & s) $\implies (x,y)$ (same argument)

On the right $(2^{2k+1}, a^2) = (r,s) = 1$ because a is odd, so $(x,y)=1$ (same argument as above!).