Math 445 HW #8 Solutions

33. Three solutions to $x^2+3y^2=7z^2$;
Find rational solutions to $\left(\frac{x}{z}\right)^2+3\left(\frac{y}{z}\right)^2=7$, i.e. $X^2+3Y^2=7$

One solution (by inspection) is $X=2, Y=1$. To find others, set $Y=r(X-2)+1$ with $r\in\mathbb{Q}$. Plugging in,

$$X^2+3(r(X-2)+1)^2=7=X^2+3r^2(X-2)^2+6r(X-2)+3,\text{ so}$$
$$(X^2-4)+(X-2)(3r^2(X-2)+6r)=0 \text{ so}$$
$$(X-2)((X+2)+3r^2(X-2)+6r)=0 \text{ so } X=2 \text{ or}$$
$$(1+3r^2)X+2-6r^2+6r=0, \text{ so } X=\frac{6r^2-6r-2}{3r^2+1}$$

Then
$$Y=r(X-2)+1=r\left(\frac{6r^2-6r-2}{3r^2+1}-2\right)+1$$
$$=r\left(\frac{6r^2-6r-2-6r^2-2}{3r^2+1}\right)+1=\frac{-6r^2-4r+3r^2+1}{3r^2+1}$$
$$=\frac{1-4r-3r^2}{3r^2+1}$$

Setting $r=\frac{-a}{b}$ (we will want $a,b>0$ so that $x,y,z>0$!)

$$\frac{x}{z}=X=\frac{6a^2+6ab-2b^2}{3a^2+b^2},\quad Y=\frac{b^2+4ab-3a^2}{3a^2+b^2}$$

For example, with $a=b=1$, we get
$\frac{x}{z}=\frac{10}{4}=\frac{5}{2}$ $\frac{y}{z}=\frac{2}{4}=\frac{1}{2}$, so $\boxed{x=5,\ y=1,\ z=2}$ works;
$25+3=28\checkmark$

with $a=1, b=2$ we get
$\frac{x}{z}=\frac{10}{7}$, $\frac{y}{z}=\frac{9}{7}$, so $\boxed{x=10,\ y=9,\ z=7}$ works
$100+243=343\checkmark$

Our original solution is $\boxed{x=2, y=1, z=1}$.

$4+3=7 \checkmark$

If I get bored, I will write a bunch more solutions after the rest of the problems.

34. Show that $4x^2+11y^3=29$ has no integer solutions.

First try working mod $\underline{4}$! $\quad 11y^3 \underset{4}{\equiv} 4x^2+11y^3 \underset{4}{\equiv} 29 \underset{4}{\equiv} 1$

Then $3 \cdot 11y^3 \underset{4}{\equiv} 33y^3 \equiv y^3 \underset{4}{\equiv} 3 \cdot 1 = 3$. $\underline{\text{But}}$ $y=3$ works. Oops.

Try working mod 11!

$$4x^2 \underset{11}{\equiv} 4x^2+11y^3 \underset{11}{\equiv} 29 \underset{11}{\equiv} 7$$

$\underline{\text{Then}}$ $3 \cdot 4x^2 \underset{11}{\equiv} 12x^2 \underset{11}{\equiv} x^2 \underset{11}{\equiv} 3 \cdot 7 \underset{11}{\equiv} 21 \underset{11}{\equiv} 10 \underset{11}{\equiv} -1$.

$\underline{\text{But}}$ $x^2 \underset{11}{\equiv} -1$ has no solutions; Euler's Criterion says to see if (since 11 is prime)

$$(-1)^{\frac{11-1}{2}} \underset{11}{\equiv} 1. \quad \text{But} \quad (-1)^{\frac{11-1}{2}} = (-1)^5 = -1 \underset{11}{\not\equiv} 1$$

$\underline{\text{Or}}$: just check:

$0^2 \underset{11}{\equiv} 0$, $1^2 \underset{11}{\equiv} 1$, $2^2 \underset{11}{\equiv} 4$, $3^2 \underset{11}{\equiv} 9$, $4^2 \underset{11}{\equiv} 16 \underset{11}{\equiv} 5$, $5^2 \underset{11}{\equiv} 25 \underset{11}{\equiv} 3$,

$6^2 \underset{11}{\equiv} 36 \underset{11}{\equiv} 3$, $7^2 \underset{11}{\equiv} 49 \underset{11}{\equiv} 5$, $8^2 \underset{11}{\equiv} 64 \underset{11}{\equiv} 9$, $9^2 \underset{11}{\equiv} 81 \underset{11}{\equiv} 4$, $10^2 \underset{11}{\equiv} 100 \underset{11}{\equiv} 1$

Note that we really only need to compute x^2 up to $\lfloor \frac{p}{2} \rfloor$; after that the resulting values repeat themselves (in the opposite order).

35. Show that $57x^2+113y^2=116z^2$ has no solution with $\boxed{xyz\neq 0}$

The only coefficient that is prime is 113.
($\underset{2}{\equiv}1$, $\underset{3}{\equiv}2$, $\underset{5}{\equiv}3$, $\underset{7}{\equiv}1$, so no prime $<\sqrt{113}$ is a factor.)

Consider the equation mod 113:

$57x^2 \underset{113}{\equiv} 57x^2+113y^2 \underset{113}{\equiv} 116z^2 \underset{113}{\equiv} 3z^2$. Then

$2\cdot 57x^2 \underset{113}{\equiv} 114x^2 \underset{113}{\equiv} x^2 \underset{113}{\equiv} 2\cdot 3z^2 \underset{114}{\equiv} 6z^2$.

Note that $113|z \Rightarrow 113|x$, so $113y^2 = 116z^2-57x^2$ is a multiple of $(113)^2$, so $113|y^2$ so $113|y$, so ~~...~~ (x,y,z) is not a primitive solution. But if the equation has a solution, then (dividing x,y,z by (x,y)) it has a primitive one. So we may assume that $z \underset{113}{\not\equiv} 0$.

But then there is a $\bar{z}$ with $\bar{z}z \underset{113}{\equiv} 1$ ($\bar{z}=z^{112}$ works!)

So $\bar{z}^2x^2 = (\bar{z}x)^2 \underset{13}{\equiv} 6\bar{z}^2z^2 \underset{113}{\equiv} 6(z\bar{z})^2 \underset{113}{\equiv} 6(1)^2 \equiv 6$.

So 6 is a square mod 113.
So 6 is a square mod 113. $a^2 \underset{113}{\equiv} 6$ has a solution $\Longleftrightarrow$

But it isn't! $\quad 56 = 32+24 = 32+16+8$

$6^{\frac{113-1}{2}} = 6^{56} \underset{113}{\equiv} 1$. But

and $6^1\equiv 6$, $6^2\equiv 36$, $6^4 \equiv (36)^2 = 1296 \equiv 11\cdot 113+53 \equiv 53$

$6^8 \equiv 53^2 = 2809 = 113\cdot 24 + 97 \equiv 97$

$6^{16} \equiv 97^2 = 9409 = 113\cdot 83 + 30 \equiv 30$

$6^{32} \equiv 30^2 = 900 = 113\cdot 8 - 4 \equiv -4$

So $6^{56} \underset{113}{\equiv} 97\cdot 30\cdot(-4) \equiv -(120)(97) \equiv -11640 = -(113\cdot 103 + 1)$

$\underset{113}{\equiv} -1 \underset{113}{\not\equiv} 1$

So 6 is <u>not</u> a square mod 113. ∎

36. If $x^2+y^2+z^2 = 2xyz$ then $x=y=z=0$.

Note that if <u>one</u> of them is $=0$, then $x^2+y^2+z^2 \equiv 0$ which implies <u>all</u> of them are $=0$. So suppose <u>none</u> of x,y,z are 0. Note that $x^2+y^2+z^2$ <u>even</u> $\Longrightarrow$ either <u>zero</u> or <u>two</u> of x,y,z are odd; in particular, at least one is <u>even</u>, so $x^2+y^2+z^2 = 2xyz \underset{4}{\equiv} 0$. But if, say, x is even and y,z are <u>odd</u>, then $x^2 \underset{4}{\equiv} 0$, $y^2 \underset{4}{\equiv} z^2 \underset{4}{\equiv} 1$, so $x^2+y^2+z^2 \underset{4}{\equiv} 2$. So it can't be the case that any of x,y,z are odd, ie, they are all even. So set $x=2x_1$, $y=2y_1$, $z=2z_1$ then we have

$4x_1^2+4y_1^2+4z_1^2 = 2(8x_1y_1z_1)$, ie. $x_1^2+y_1^2+z_1^2 = 4x_1y_1z_1$.

But considering <u>this</u> equation mod 4 will again lead us to conclude that $x_1 = 2x_2$, $y_1 = 2y_2$, $z_1 = 2z_2$, and then

$x_2^2+y_2^2+z_2^2 = \underline{8}\,x_2y_2z_2$.

This cannot continue! One way to make this precise is to consider the equations $x^2+y^2+z^2 = 2^k xyz$, $k \geq 1$, together. If we choose the $k \geq 1$ with the solution $x,y,z \geq 1$ with smallest x (say), then our argument above implies that $x = 2x_1, y = 2y_1, z = 2z_1$ with $x_1^2+y_1^2+z_1^2 = 2^{k+1}x_1y_1z_1$ and $x_1 < x$, a contradiction. So none of the equations can have a solution with $x,y,z \geq 1$. So

$$x^2+y^2+z^2 = 2xyz \implies x=y=z=0. \quad ///$$

37. p prime, then $(x^2-17)(x^2-19)(x^2-323) \equiv_p 0$ has a solution.

If $p=2$, Take $x=1$: $1^2-17 = -16 \equiv_{2} 0$, so the product will be $\equiv_2 0$.

If $p>2$, then p is odd, and we know by Euler's criterion that $x^2 \equiv_p a$ has a solution $\iff$ $a^{\frac{p-1}{2}} \equiv_p 1$. But since $a^{p-1} \equiv_p 1$, we know that $a^{\frac{p-1}{2}} \equiv_p 1$ or -1. So if $x^2 \equiv_p 17$ has no solution and $x^2 \equiv_p 19$ has no solution, then $17^{\frac{p-1}{2}} \equiv_p -1$, and $19^{\frac{p-1}{2}} \equiv_p -1$. But then $323^{\frac{p-1}{2}} = (17\cdot 19)^{\frac{p-1}{2}} = 17^{\frac{p-1}{2}} \cdot 19^{\frac{p-1}{2}} \equiv_p (-1)(-1) = 1$, so $x^2 \equiv_p 323$ has a solution. So for every p, there is an x so that either $p \mid x^2-17$ or $p \mid x^2-19$ or $p \mid x^2-323$, so there is an x so that $p \mid (x^2-17)(x^2-19)(x^2-323)$, ie,

$$(x^2-17)(x^2-19)(x^2-323) \equiv_p 0 \text{ has a solution. } ///$$

#33, continued.

I wasn't really bored, but you folks came up with a lot of different solutions. Among them:

(5, 1, 2), (34, 3, 13), (88, 1, 31), (10, 9, 7),
(10, 47, 31), (10, 19, 13), (59, 47, 38), (17, 19, 14),
(1, 3, 2), (125, 87, 74), (86, 27, 37), (50, 3, 19)

Just for fun, I decided to try $a = 47$ $b = 23$, which gives (9341, 887, 3578) (after dividing by the gcd).