Math 445
Handy facts for the second exam

Don't forget the handy facts from the first exam!

Continued Fractions.

If we look at each line of the calculation of g.c.d of a and b,

$$a = bq_0 + r_0,\ b = r_0q_1 + r_1,\ \ldots,\ r_{n-2} = r_{n-1}q_n + r_n,\ r_n = r_{n-1}q_{n+1} + 0$$

they can we re-written as

$$\frac{a}{b} = q_0 + \frac{r_0}{b}, \frac{b}{r_0} = q_1 + \frac{r_1}{r_0}, \ldots \frac{r_{n-2}}{r_{n-1}} = q_n + \frac{r_n}{r_{n-1}}, \frac{r_n}{r_{n-1}} = q_{n+1}$$

When we put these together, we get a *continued fraction expansion* of a/b

$$(*) \qquad \frac{a}{b} = q_0 + \cfrac{1}{q_1 + \cfrac{1}{q_2 + \cfrac{1}{\ldots + \cfrac{1}{q_{n+1}}}}}$$

which, for the sake of saving space, we will denote $\langle q_0, q_1, \ldots, q_{n+1}\rangle$. Note that, conversely, given a collection $q_0, \ldots, q_{n+1}$ of integers, we can construct a rational number, which we denote $\langle q_0, q_1, \ldots, q_{n+1}\rangle$ by the formula (*).

Formally, we can try to do the same thing with any real number x; i.e, "compute" the g.c.d. of x and 1 :

$$x = a_0 + r_0,\ 1 = r_0a_1 + r_1,\ \ldots,\ r_{n-2} = r_{n-1}a_n + r_n$$

Unlike for the rational number a/b, if x is irrational, we shall see that this process does not terminate, giving us an "infinite" continued fraction expansion of x, $\langle a_0, a_1, a_2 \ldots\rangle$. Our main goal is to figure out what this sequence of integers means!

First, a slightly different perspective:

$x = q_0 + r_0$ with $0 \leq r_0 < 1$ means $q_0 = \lfloor x\rfloor$ is the largest integer $\leq x$; $\lfloor \text{blah}\rfloor$ is the *greatest integer function.* $1 = r_0q_1 + r_1$ with $0 \leq r_1 < r_0$ means $1/r_0 = q_1 + (r_1/r_0) = q_1 + x_1$ with $0 \leq x_1 < 1$, so $q_1 = \lfloor 1/r_0\rfloor$. In general, the process of extracting the continued fraction exapnsion of x looks like:

(**) $\qquad x = \lfloor x\rfloor + x_0 = a_0 + x_0, 1/x_0 = \lfloor 1/x_0\rfloor + x_1 = a_1 + x_1, \ldots,$
$1/x_{n-1} = \lfloor 1/x_{n-1}\rfloor + x_n = a_n + x_n, \ldots$

If we stop this at any finite stage, then we can, just as in the case of a rational number a/b, reassemble the pieces to give

$$x = \langle a_0, a_1, \ldots, a_{n-1}, a_n + x_n\rangle = \langle a_0, a_1, \ldots, a_{n-1}, a_n, 1/x_n\rangle$$

If we ignore the last x_n, we find that $\{a_0, a_1, \ldots, a_{n-1}, a_n\}$ is a rational number (proof: induction on n), called the n^{th} *convergent* of x. The integers a_n are called the n^{th} *partial quotients* of x. Note that since $0 \leq x_0 < 1$, $1/x_0 > 1$, so $a_1/geq1$. This is true for all later calculations, so $a_i \geq 1$ for all $i/geq1$. This sort of continued fraction expansion is what is called *simple.* We will, in our studies, only deal with simple continued fractions.

For example, we can compute that, for $x = \sqrt{2}$, $a_0 = 1$, $x_0 = \sqrt{2} - 1$, $1/x_0 = \sqrt{2} + 1$, $a_1 = 2$, $x_1 = \sqrt{2} - 1 = x_0$, so the pattern will repeat, and $\sqrt{2}$ has continued fraction expansion $lcf1, 2, 2, \ldots\rangle$. By computing some partial quotients, one can show that π has expansion that begins $\langle 3, 7, \rangle$

By looking at the expression for a continued fraction, that we started with, it should be apparent that

$$\langle a_0, a_1, \ldots, a_{n-1}, a_n\rangle = \langle a_0, a_1, \ldots, a_{n-1} + \frac{1}{a_n}\rangle = a_0 + \frac{1}{\langle a_1, \ldots, a_{n-1}, a_n\rangle}$$

From this it follows, for example, that $\langle a_0, a_1, \ldots, a_{n-1}, a_n\rangle = \langle a_0, a_1, \ldots, a_{n-1}, a_n - 1, 1\rangle$. But these are the only such equalities:

Prop: If $\langle a_0, a_1, \ldots, a_n\rangle = \langle b_0, b_1, \ldots, b_m\rangle$ and $a_n, b_m > 1$, then $n = m$ and $a_i = b_i$ for all $i = 0, \ldots, n$.

Computing $\langle a_0, a_1, \ldots, a_n\rangle$ from $\langle a_0, a_1, \ldots, a_{n-1}\rangle$:

$\langle a_0, a_1, \ldots, a_n\rangle = \dfrac{h_n}{k_n}$, where $h_{-2} = 0, k_{-2} = 0, h_{-1} = 1, k_{-1} = 0$, and for $i \geq 0$,

$$h_i = a_i h_{i-1} + h_{i-2} \text{ and } k_i = a_i k_{i-1} + k_{i-2}.$$

The proof is by induction. This, in turn implies:

For every $i \geq 0$, $h_i k_{i-1} - h_{i-1} k_i = (-1)^{i-1}$ (which implies that $(h_i, k_i) = 1$), and $h_i k_{i-2} - h_{i-2} k_i = (-1)^i a_i$.

Note: None of these formulas actually require that the a_i's are integers.

for $x = \langle a_0, a_1, \ldots, a_{n-1}, a_n + x_n\rangle = \langle a_0, a_1, \ldots, a_{n-1}, a_n, \dfrac{1}{x_n}\rangle$, if we set

$$\langle a_0, a_1, \ldots, a_{n-1}, a_n\rangle = r_n,$$

then these formulas imply that

$$r_{2n} < r_{2n+2} \text{ and } r_{2n-1} > r_{2n+1} \text{ for every } n, \text{ and } r_{2n} - r_{2n-1} = \frac{1}{k_{2n-1}k_{2n}}$$

And since the numerator of

$x - \langle a_0, a_1, \ldots, a_{n-1}, a_n\rangle = \langle a_0, a_1, \ldots, a_{n-1}, a_n + x_n\rangle - \langle a_0, a_1, \ldots, a_{n-1}, a_n\rangle$,

we can compute, is $x_n(h_{n-1}k_{n-2} - h_{n-2}k_{n_1})$ (and the denomenator is positive), we have that $r_{2n} < x < r_{2n+1}$. So since $r_{2n} - r_{2n-1} \to 0$ as $n \to \infty$, we find that $r_n \to x$, In particular, $|x - r_n| < |r_{n-1} - r_n| = 1/(k_{n-1}k_n)$ for every n. This implies that if the x_n are never 0 (i.e., the continued fraction process is really an infinite one), then since $0 < |k_n(x - r_n)| = |k_n x - h_n| < 1/k_{n-1}$, we find that x is not rational.

Note that since $a_i \geq 1$ for every $i > 0$, $k_{-1} = 0, k_0 = 1$, and $k_i = a_i k_{i-1} + k_{i-2} \geq k_{i-1} + k_{i-2}$ for every $i \geq 1$, we can see by induction that $k_n \geq$ the n^{th} Fibonacci number (which is defined by $F_i = F_{i-1} + F_{i-2}$), and the Fibonacci numbers grow very fast!

Based on these facts, we denote $x = \lim_{n\to\infty} \langle a_0, \ldots, a_n\rangle = \langle a_0, a_1, \ldots\rangle$. Then

$$\langle a_0, a_1, \ldots\rangle = a_0 + \frac{1}{\langle a_1, a_2, \ldots\rangle}$$

which in turn implies that:

If $\langle a_0, a_1, \ldots\rangle = \langle b_0, b_1, \ldots\rangle$, then $a_i = b_i$ for all i.

If $1 \leq b < k_n$, then $|x - \dfrac{a}{b}| \geq |x - \dfrac{h_n}{k_n}|$ for all integers a; in fact if $1 \leq b < k_{n+1}$, then $|bx - a| \geq |k_n x - h_n|$ for all integers a.

If $x \notin \mathbb{Q}$ and $a, b \in \mathbb{Z}$, with $|x - \dfrac{a}{b}| < \dfrac{1}{2b^2}$, then $\dfrac{a}{b} = \dfrac{h_n}{k_n}$ for some n.

Repeating continued fraction expansions: A continued fraction $\langle a_0, a_1, \ldots\rangle$ will repeat (i.e, $a_n = a_{n+m}$ for all $n \geq N$) precisely when $x_{n-1} = x_{n+m-1}$, since from (**) above, all of the calculations of the partial quotients, starting from some fixed number, will depend only on that fixed number. A real number x has a repeating continued fraction expansion if and only if x is an (irrational) root of a quadratic equation, what we call a *quadratic irrational*. In particular,

For any non-square positive integer n, $\sqrt{n} + \lfloor\sqrt{n}\rfloor = \langle\overline{2a_0, a_1, \ldots a_m}\rangle$ is *purely periodic*. This implies that $\sqrt{n} = \langle a_0, \overline{a_1, \ldots a_m, 2a_0}\rangle$

Pell's Equation.

It turns out that the continued fraction expansion of $\sqrt{n}$ can help us find the integer solutions x, y of the equation

$$(***) \qquad x^2 - ny^2 = N$$

for fixed values of n and N. This equation is known as *Pell's equation*.

First the less interesting cases. If $n < 0$, then any solution to $N = x^2 - ny^2 \geq x^2 + y^2$ has $|x|, |y| \leq \sqrt{N}$, which can be found by inspection. If $n = m^2$ for some m, then $N = x^2 - m^2y^2 = (x - my)(x + my)$, so $x - my, x + my$ both divide N, so, e.g., their sum, $2x$ divides N^2. We can then find all possible x, and so all solutions, by inspection. We now focus on finding solutions for $n \geq 1$ not a perfect square. $\sqrt{n}$ is therefore irrational.

Then if $1 \leq N \leq \sqrt{n}$ is not a perfect square, then $N = x^2 - ny^2$ implies that

$|\sqrt{n} - \frac{x}{y}| = \frac{N}{|x + \sqrt{n}y| \cdot |y|} < \frac{N}{2\sqrt{n}y^2} < \frac{1}{2y^2}$, so $\frac{x}{y} = \frac{h_m}{k_m}$ for some m.

(The same, it turns out, is true for $-\sqrt{n} \leq N \leq -1$.) But which m?

$\sqrt{n} = \langle a_0, \overline{a_1, \ldots a_m, 2a_0}\rangle$ means that $\sqrt{n} = \langle a_0, a_1, \ldots a_m, a_0 + \sqrt{n}\rangle$. In general, at any point where we stop computing the continued fraction of $\sqrt{n}$, we find that

$\sqrt{n} = \langle b_0, b_1, \ldots b_s, \frac{\sqrt{n}+a}{b}\rangle$, where $\frac{1}{x_s} = \frac{\sqrt{n}+a}{b}$

(so a and b take on only finitely many values, because x_s does). But then we can compute that

$\sqrt{n} = \frac{(\frac{\sqrt{n}+a}{b})h_s + h_{s-1}}{(\frac{\sqrt{n}+a}{b})k_s + k_{s-1}}$, which implies that $h_s^2 - nk_s^2 = b(h_sk_{s-1} - h_{s-1}k_s) = (-1)^{s-1}b$.

In particular, solutions to $x^2 - ny^2 = 1$ exist, because $b = 1$ occurs as the denominator of x_i for $i = m + 1, 2m + 1, 3m + 1, \ldots$. These are either all odd (if m is even), or every other one is odd. For these values, $i - 1$ is even, so $h_i^2 - nk_i^2 = b(h_ik_{i-1} - h_{i-1}k_i) = (-1)^{i-1}b = 1$.

There is an alternative approach to generating solutions to (***). If we know that $x^2 - ny^2 = N$ and $x_0^2 - ny_0^2 = 1$, then

$$(x^2 - ny^2)(x_0^2 - ny_0^2)^m = N = (x - \sqrt{n}y)(x_0 - \sqrt{n}y_0)^m(x + \sqrt{n}y)(x_0 + \sqrt{n}y_0)^m$$

But $(x^2 - ny^2)(x_0^2 - ny_0^2)^m = A - \sqrt{n}B$ for some A, B, and then $(x^2 + ny^2)(x_0^2 + ny_0^2)^m = A + \sqrt{n}B$ (because of the properties of *conjugates* of quadratic irrationals). Then $(A - \sqrt{n}B)(A - +\sqrt{n}B) = A^2 - nB^2 = N$.

Diophantine Equations.

Equations like $x^2 - 17y^2 = 3$, for which we seek solutions with $x, y \in \mathbb{Z}$ form a class of equations called *Diophantine Equations.* Typically, we have two goals: decide if the equation has a solution; if it does, then we wish to describe all of the solutions.

In principle, a Diophantine equation may really be a system of equations:
$f_1(x_1, \dots, x_n) = 0, \dots, f_m(x_1, \dots, x_n) = 0$; in theory, these can be replaced by one equation $[f_1(x_1, \dots, x_n)]^2 + \cdots [f_m(x_1, \dots, x_n)]^2 = 0$, although this rarely makes finding a solution easier!

For example, by the Euclidean algorithm, the Diophantine equation

$$ax + by = c$$

has a solution $\Leftrightarrow (a, b)|c$. The Euclidean algorithm will provide a solution to $ax_0 + bx_0 = (a, b)$; then if $a = a_0(a, b)$, $b = b_0(a, b)$, $c = c_0(a, b)$, then the solutions to $ax + by = c$ are $x = c_0x_0 + nb_0, y = c_0y_0 - na_0$ for $n \in \mathbb{Z}$.

As another example, for the equation $ax^2 + by = c$ to have a solution, $aX + bY = c$ must; so we need $(a, b)|c$. But this is in general not sufficient; treating the original equation mod b, we need $ax^2 \equiv c \pmod{b}$, which may not have a solution. If $aA \equiv 1 \pmod{b}$, for example, then we need Ac to be a square, mod b; Euler's criterion can help us decide if it is.

Pythagorean triples: Solutions to $x^2 + y^2 = z^2$. If (x, y, z) is a *Pythagorean triple*, then if $(x, y) = d > 1$ then $d|z$, as well, so $(x/d)^2 + (y/d)^2 = (z/d)^2$ is a solution, as well. We therefore look for *primitive solutions*, i.e., those with $(x, y) = (y, z) = (x, z) = 1$. BY looking at the equation mod 4, we can see that z must be odd, and x and y have opposite parity; let us assume that x is even. Then by rewriting the equation as $x = 2u$, and $x^2 = z^2 - y^2 = (z + y)(z - y)$, we find that
$u^2 = (\frac{z+y}{2})(\frac{z-y}{2})$; but $(\frac{z+y}{2}, \frac{z-y}{2}) = 1$, so each must be a perfect square r^2, s^2, implying that $z = r^2 + s^2$, $y = r^2 - s^2$, and $x = 2rs$. (Note that r and$s$ must have opposite parity, so that y and z are odd.) Conversely, we can compute that such values of x, y, z satisfy $x^2 + y^2 + z^2$, so

$(x, y, z) = (2rs, r^2 - s^2, r^2 + s^2)$, $(r, s) = 1$, $r - s$ odd, gives all primite Pythagorean triples.

The above argument used: $(a, b) = 1$ and $ab = c^2$ implies $a = u^2, b = v^2$ for some u, v .

By contrast, the equation $x^4 + y^4 = z^2$ has no solution with $x, y, z \in \mathbb{Z}$ and $xyz \neq 0$; consequently, $x^4 + y^4 = z^4$ also has no solutions.

Local versus global solutions.

If the equation $f(x_1, \dots, x_n) = 0$ has a solution with $x_i \in \mathbb{Z}$ for all i, then it is certainly the case that $f(x_1, \dots, x_n) = 0$ has a solution with $x_i \in \mathbb{R}$ for all i (use the same solution!). Similarly, the equation $f(x_1, \dots, x_n) \equiv 0$ (mod N has a solution for every N. Solutions to these latter equations are called *local* solutions; by analogy, a solution to our original Diophantine equation is then called a *global* solution. This implies that if we can show that an diophantine equation has no local solution for some n or for $\mathbb{R}$, then the original equation has no global solution.

For example, by working mod 5, we can show that the equation $2x^2 + 5y^2 = 9z^2$, since it has no primitive solutions. Any such primitive solution would also solve $x^2 \equiv 27z^2 \equiv 2z^2$. If

$5|z$ then $5|x$, so $25|5y^2$, so $5|y$, and we do not have a primitive solution. Then we may invert z mod 5; finding w with $zW \equiv 1 \pmod 5$ anbd multiplying both sides of our equation with w^2, we get $(xw)^2 \equiv 2 \pmod 5$; but a quick check of all representatives mod 5 (like 1,2,3,4), or using Euler's criterion, we find that 2 is not a square mod 5.

There are, however, equations which have all types of local solutions, but no global one; the first such equation found was $x^4 - 17 = 2y^2$.

Geometric solutions.

For equations such as $x^2 + 10y^2 = 19z^2$ where we know one solution (like (3,1,1)), we can find all solutions using a geometric process. Setting $X = x/z$, $Y = y/z$, our equation becomes

$$(****) \qquad X^2 + 10Y^2 = 19 \text{ (in this case, an ellipse)}$$

for which we know one (rational) solution; (3,1). Our goal is now to find all *rational* solutions (the denomenator will be our z). But if we imagine having another rational solution (a, b), then the line through $(3, 1)(in our case) and (a, b)$ will have rational slope. If we take the equation for this line and plug it into (****), we get a quadratic equation with (because of the rational slope) rational coefficients, for which we know one, rational, solution (in our case, $X = 3$). The other solution must therefore be rational, and the corresponding point on the line then has rational coordinates. In our example, this procedure looks like

$Y = r(X-3)+1$, so $x^2+10(r(X-3)+1)^2 = 19$, i.e., $(X^2-9)+10r^2(X-3)^2+20r(X-3) = 0$, i.e., $(X-3)(X+3+10r^2X-30r^2+20r) = 0$. So $X = 3$ or $(10r^2+1)X-(30r^2-20r-3) = 0$, i.e., (setting $r = a/b$)

$$X = \frac{30r^2 - 20r - 3}{10r^2 + 1} = \frac{30a^2 - 20ab - 3b^2}{10a^2 + b^2}$$

so $x = 30a^2 - 20ab - 3b^2$, $z = 10a^2 + b^2$ and (by plugging into the equation for the line) $y = -(10a^2 + 6ab - b^2)$ provide solutions.

Sums of four squares.

For every $n \in \mathbb{N}$, there are $x, y, z, w \in \mathbb{Z}$ so that $x^2 + y^2 + z^2 + w^2 = n$.

Elements of the proof:

$$(x_1^2 + y_1^2 + z_1^2 + w_1^2)(x_2^2 + y_2^2 + z_2^2 + w_2^2) =$$
$$(x_1x_2 + y_1y_2 + z_1z_2 + w_1w_2)^2 + (x_1y_2 - x_2y_1 + z_2w_1 - z_1w_2)^2 +$$
$$(x_1z_2 - x_2z_1 + y_1w_2 - w_1y_2)^2 + (x_1w_2 - x_2w_1 + y_2z_1 - y_1z_2)^2$$

so we may focus on primes p. $p = 2 = 1^2 + 1^2 + 0^2 + 0^2$, so focus on odd primes. Then

$0 \le x, y \le (p-1)/2$ and $x \ne y$ implies $x^2 \not\equiv y^2 \pmod p$, so for any a, x^2 and $a - y^2$, with $0 \le x, y \le (p-1)/2$ must have a value, mod p, in common (otherwise $x^2 + y^2 - a$ takes on $p + 1$ different values, mod p). So $x^2 + y^2 \equiv -1 \pmod p$ has a solution. Then $x^2 + y^2 + 1^2 + 0^2 = Mp$ for some M; with the restrictions on x, y, we have $M < p$. Choose the smallest positive M with $Mp = x^2 + y^2 + z^2 + w^2$. M is odd, since otherwise (after renaming the variables to group them by parity)

$$\frac{M}{2}p = (\frac{x-y}{2})^2 + (\frac{x+y}{2})^2 + (\frac{z-w}{2})^2 + (\frac{z+w}{2})^2$$

If $M > 1$, then choose $-\frac{M}{2} \le x_1, y_1, z_1, w_1 \le \frac{M}{2}$ with $x \equiv x_1 \pmod M$, etc. then

$x_1^2 + y_1^2 + z_1^2 + w_1^2 \equiv x^2 + y^2 + z^2 + w^2 \equiv 0 \pmod{M}$, so $x_1^2 + y_1^2 + z_1^2 + w_1^2 = NM$ with (from the restrictions on x_1, etc.) $N < M$. Then

$NM^2 p = (x_1^2 + y_1^2 + z_1^2 + w_1^2)(x^2 + y^2 + z^2 + w^2)$ = a sum of four squares with, we can compute, every term a multiple of M! Dividing through by M^2, we find that Np is a sum of four squares, with $N < M$, contradicting the choice of M. So $M = 1$, and we are done.