Math 445 Number Theory

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Theorem: If p is prime, the equation $x^2 \equiv -1 \pmod{p}$ has a solution $\Leftrightarrow p = 2$ or $p \equiv 1 \pmod 4$. Last time did $\Leftarrow$; now we do: If $p \equiv 3 \pmod 4$ is prime, then $x^2 \equiv -1 \pmod p$ has no solution. This is really rather quick. If $x^2 \equiv -1 \pmod p$, then since by FLT $x^{p-1} \equiv 1 \pmod p$, we have, mod p,

$1 \equiv x^{p-1} = x^{(4k+3)-1} = x^{4k+2} = x^{2(2k+1)} = (x^2)^{2k+1} \equiv (-1)^{2k+1} = -1$, so $1 \equiv -1 \pmod p$. i.e., $p|2$, which is absurd.

With this in hand, we can show: *Proposition:* If $n = a^2 + b^2$, $p|n$, and $p \equiv 3 \pmod 4$, then $p|a$ and $p|b$.

If not, then either $p \not| a$ or $p \not| b$, say $p \not| a$. Then $(a,p) = 1$, so there is a z with $az \equiv 1 \pmod p$. But then since $p|n$, $p|a^2+b^2$, so $a^2+b^2 \equiv 0 \pmod p$. Then $1 + (bz)^2 = (az)^2 + (bz)^2 = z^2(a^2+b^2) \equiv z^2 0 = 0 \pmod p$, so $x = bz$ satisfies $x^2 + 1 \equiv 0 \pmod p$, i.e., $x^2 \equiv -1 \pmod p$, a contradication. So $p|a$ and $p|b$.

(*) This means that $p^2|a^2$ and $p^2|b^2$, so $p^2|a^2+b^2 = n$, and $(n/p^2) = (a/p)^2 + (b/2p)^2$. This will be very significant shortly! The final peice of the puzzle is:

Proposition: If $p \equiv 1 \pmod 4$ and p is prime, then $p = a^2 + b^2$ for some integers a , b .

To see this, set $k = \lfloor \sqrt{p} \rfloor$ = the largest integer $\leq p$. Since p is prime, $\sqrt{p}$ is not an integer, so $k < \sqrt{p} < k+1$. Because $p \equiv 1 \pmod 4$, there is an x with $x^2 \equiv -1 \pmod p$. Now look at the collection of integers $u + xv$ for $0 \leq u \leq k$ and $0 \leq v \leq k$. Since there are $(k+1)^2 > p$ of them, at least two of them are congruent mod p; $u_1 + xv_1 \equiv u_2 + xv_2$. Then $u_1 - u_2 \equiv xv_2 - xv_1 = x(v_2 - v_1)$, so $(u_1-u_2)^2 \equiv x^2(v_2-v_1)^2 = -(v_2-v_1)^2$. Setting $a = u_1 - u_2$ and $b = v_2 - v_1$, this means $p|a^2+b^2$. But since either $u_1 \neq u_2$ or $v_1 \neq v_2$, $a^2+b^2 > 0$. Also, since $0 \leq u_1, u_2, v_1, v_2 \leq k$, $|u_1-u_2|, |v_2-v_1| \leq k$, so $a^2+b^2 \leq k^2+k^2 = 2k^2 < 2p$. So $0 < a^2+b^2 < 2p$ and is divisible by p ; so $a^2+b^2 = p$, as desired.

So now we know that (1) the product of two sums of two squares is a sum of two squares, (2) 2 and any prime $\equiv 1 \pmod 4$ is a sum of two squares, and (3) and prime $\equiv 3 \pmod 4$ which divides a^2+b^2 divides both a and b. Putting these together, we can completely characterize which numbers can be expressed as a^2+b^2 :

Theorem: If $n = 2^k p_1^{k_1} \cdots p_r^{k_r} q_1^{m_1} \cdots q_s^{m_s}$ is the prime factorization of n, where $p_i \equiv 1 \pmod 4$ and $q_i \equiv 3 \pmod 4$ for every i , then $n = a^2+b^2$ for some integers $a, b \Leftrightarrow m_i$ is even for every i .

The idea: use (*) above to show that if $n = a^2+b^2$ then each of the primes q_i can be divided out two at a time as $(n/q_i^2) = (a/q_i)^2 + (b/q_i)^2$, until there are none left, showing that their exponents are all even. Conversely, (by induction) $2^k p_1^{k_1} \cdots p_r^{k_r}$ is a sum of two squares, since each factor is, and then since the remaining factor $q_1^{m_1} \cdots q_s^{m_s} = q_1^{2u_1} \cdots q_s^{2u_s} = (q_1^{u_1} \cdots q_s^{u_s})^2 + 0^2$ is a sum of squares, the product, n , is a sum of two squares.

So, for example, since we know $p = 61 \cdot 2^{285652} + 1$ is prime and (as one of our class members pointed out!) $4|2^{285652}$ so $p \equiv 1 \pmod 4$, this number <u>can</u> be expressed as the sum of two squares. Care to figure out which ones?