

Math 445 Number Theory

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Proposition: If f is a polynomial with integer coefficients and $(M, N) = 1$, then the congruence equation $f(x) \equiv 0 \pmod{MN}$ has a solution \Leftrightarrow the equations $f(x) \equiv 0 \pmod{M}$ and $f(x) \equiv 0 \pmod{N}$ both do.

The direction (\Rightarrow) is immediate; $MN|f(x)$ implies $M|f(x)$ and $N|f(x)$, since $M, N|MN$. The point to (\Leftarrow) is that the solutions we know of to each of the two equations might be *different*: $f(x_1) \equiv 0 \pmod{M}$ and $f(x_2) \equiv 0 \pmod{N}$. What we wish to show is that a single number solves *both*, since then $M|f(x_0)$ and $N|f(x_0)$, and then $(M, N) = 1$ implies that $MN|f(x_0)$.

To do this, we use the fact that f is a polynomial, since then if $a \equiv b \pmod{n}$, then $f(a) \equiv f(b) \pmod{n}$. So if we suppose that we have found a and b with $f(a) \equiv 0 \pmod{M}$ and $f(b) \equiv 0 \pmod{N}$, then any x satisfying both $x \equiv a \pmod{M}$ and $x \equiv b \pmod{N}$ will satisfy both $f(x) \equiv 0 \pmod{M}$ and $f(x) \equiv 0 \pmod{N}$ simultaneously, as desired. So it is enough to show that for any a, b , there is an x which simultaneously satisfies

$$x \equiv a \pmod{M} \quad \text{and} \quad x \equiv b \pmod{N}$$

But since $(M, N) = 1$, this is true by the Chinese Remainder Theorem. In fact, finding x is a matter of solving $x = a + Mi$, $x = b + Nj$, so we need $a + Mi = b + Nj$, so $b - a = Mi - Nj$. But since $(M, N) = 1$, we can use the Euclidean algorithm to write $1 = MI_0 + NJ_0$, and then $i = (b - a)I_0, j = -(b - a)J_0$ will work, allowing us to solve for x . In fact, since the only other I, J which will work are $I = I_0 + kN, J = J_0 - kM$, we find that our solution x is unique modulo MN .

For any pair of solutions a, b to $f(a) \equiv 0 \pmod{M}$ and $f(b) \equiv 0 \pmod{N}$ there is a unique corresponding x mod MN (with $x \equiv a \pmod{M}$ and $x \equiv b \pmod{N}$) satisfying $f(x) \equiv 0 \pmod{MN}$. Introducing the notation $S(n) =$ the number of solutions, mod n , to the equation $f(x) \equiv 0 \pmod{n}$, we then have shown that $S(MN) = S(M)S(N)$ whenever $(M, N) = 1$. So by induction, whenever N_1, \dots, N_k are relatively prime, $S(N_1 \cdots N_k) = S(N_1) \cdots S(N_k)$.

So if $N = p_1^{k_1} \cdots p_r^{k_r}$ is the prime factorization of the odd number N , then for any $(a, N) = 1$ (so $(a, p_i) = 1$ for each i) we have $x^n \equiv a \pmod{N}$ has solutions $\Leftrightarrow x^n \equiv a \pmod{p_i^{k_i}}$ does for every i , and we know how to determine when that occurs.

Quadratic Residues: If $x^2 \equiv a \pmod{n}$ has a solution, a is a *quadratic residue* modulo n . If it doesn't, a is a *quadratic non-residue* modulo n . Euler's Criterion gives us a test: if p is a prime, then a is a quadratic residue mod $n \Leftrightarrow a^{\frac{p-1}{2}} \equiv 1 \pmod{p}$. But this may require a lot of calculation if p is large; our next task is to find a quicker way.

To talk about things in a compact manner, we introduce the *Legendre symbol*; for p an odd prime,

$$\left(\frac{a}{p}\right) = \begin{cases} 0 & \text{if } p|a \\ 1 & \text{if } a \text{ is a quadratic residue mod } p \\ -1 & \text{if } a \text{ is a quadratic non-residue mod } p \end{cases}$$

By Euler's criterion, this really means $\left(\frac{a}{p}\right) \equiv a^{\frac{p-1}{2}} \pmod{p}$, but our goal is to find a *quicker* way to compute it!