

*Proposition:* If  $f$  is a polynomial with integer coefficients and  $(M, N) = 1$ , then the congruence equation  $f(x) \equiv 0 \pmod{MN}$  has a solution  $\Leftrightarrow$  the equations  $f(x) \equiv 0 \pmod{M}$  and  $f(x) \equiv 0 \pmod{N}$  both do.

The direction  $(\Rightarrow)$  is immediate;  $MN|f(x)$  implies  $M|f(x)$  and  $N|f(x)$ , since  $M, N|MN$ . The point to  $(\Leftarrow)$  is that the solutions we know of to each of the two equations might be *different*:  $f(x_1) \equiv 0 \pmod{M}$  and  $f(x_2) \equiv 0 \pmod{N}$ . What we wish to show is that a single number solves *both*, since then  $M|f(x_0)$  and  $N|f(x_0)$ , and then  $(M, N) = 1$  implies that  $MN|f(x_0)$ .

To do this, we use the fact that  $f$  is a polynomial, since then if  $a \equiv b \pmod{n}$ , then  $f(a) \equiv f(b) \pmod{n}$ . So if we suppose that we have found  $a$  and  $b$  with  $f(a) \equiv 0 \pmod{M}$  and  $f(b) \equiv 0 \pmod{N}$ , then any  $x$  satisfying both  $x \equiv a \pmod{M}$  and  $x \equiv b \pmod{N}$  will satisfy both  $f(x) \equiv 0 \pmod{M}$  and  $f(x) \equiv 0 \pmod{N}$  simultaneously, as desired. So it is enough to show that for any  $a, b$ , there is an  $x$  which simultaneously satisfies

$$x \equiv a \pmod{M} \quad \text{and} \quad x \equiv b \pmod{N}$$

But since  $(M, N) = 1$ , this is true by the Chinese Remainder Theorem. In fact, finding  $x$  is a matter of solving  $x = a + Mi$ ,  $x = b + Nj$ , so we need  $a + Mi = b + Nj$ , so  $b - a = Mi - Nj$ . But since  $(M, N) = 1$ , we can use the Euclidean algorithm to write  $1 = MI_0 + NJ_0$ , and then  $i = (b - a)I_0$ ,  $j = -(b - a)J_0$  will work, allowing us to solve for  $x$ . In fact, since the only other  $I, J$  which will work are  $I = I_0 + kN$ ,  $J = J_0 - kM$ , we find that our solution  $x$  is unique modulo  $MN$ .

For any pair of solutions  $a, b$  to  $f(a) \equiv 0 \pmod{M}$  and  $f(b) \equiv 0 \pmod{N}$  there is a unique corresponding  $x \pmod{MN}$  (with  $x \equiv a \pmod{M}$  and  $x \equiv b \pmod{N}$ ) satisfying  $f(x) \equiv 0 \pmod{MN}$ . Introducing the notation  $S(n)$  = the number of solutions, mod  $n$ , to the equation  $f(x) \equiv 0 \pmod{n}$ , we then have shown that  $S(MN) = S(M)S(N)$  whenever  $(M, N) = 1$ . So by induction, whenever  $N_1, \dots, N_k$  are relatively prime,  $S(N_1 \cdots N_k) = S(N_1) \cdots S(N_k)$ .

So if  $N = p_1^{k_1} \cdots p_r^{k_r}$  is the prime factorization of the odd number  $N$ , then for any  $(a, N) = 1$  (so  $(a, p_i) = 1$  for each  $i$ ) we have  $x^n \equiv a \pmod{N}$  has solutions  $\Leftrightarrow x^n \equiv a \pmod{p_i^{k_i}}$  does for every  $i$ , and we know how to determine when that occurs.

**Quadratic Residues:** If  $x^2 \equiv a \pmod{n}$  has a solution,  $a$  is a *quadratic residue* modulo  $n$ . If it doesn't,  $a$  is a *quadratic non-residue* modulo  $n$ . Euler's Criterion gives us a test: if  $p$  is a prime, then  $a$  is a quadratic residue mod  $n \Leftrightarrow a^{\frac{p-1}{2}} \equiv 1 \pmod{p}$ . But this may require a lot of calculation if  $p$  is large; our next task is to find a quicker way.

To talk about things in a compact manner, we introduce the *Legendre symbol*; for  $p$  an odd prime,

$$\left(\frac{a}{p}\right) = \begin{cases} 0 & \text{if } p|a \\ 1 & \text{if } a \text{ is a quadratic residue mod } p \\ -1 & \text{if } a \text{ is a quadratic non-residue mod } p \end{cases}$$

By Euler's criterion, this really means  $\left(\frac{a}{p}\right) \equiv a^{\frac{p-1}{2}} \pmod{p}$ , but our goal is to find a *quicker* way to compute it!