Math 445 Number Theory

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From last time: $x = [a_0, a_1, \dots, a_n, \dots]$; set $r_n = [a_0, \dots, a_n] = \dfrac{h_n}{k_n}$. Then $|x - r_n| < \dfrac{1}{k_n k_{n+1}} \leq \dfrac{1}{k_n^2 a_{n+1}}$. In particular, $\dfrac{h_n}{k_n} \to x$ as $n \to \infty$.

From this, we can learn many things! First, if $x = [a_0, a_1, \dots]$ with $a_1 \geq 1$ for all $i \geq 1$, then $x \notin \mathbb{Q}$; because if $x = \dfrac{a}{b}$, then $|\dfrac{a}{b} - \dfrac{h_n}{k_n}| < \dfrac{1}{k_n k_{n+1}}$, so $|ak_n - bh_n| < \frac{|b|}{k_{n+1}} \to 0$ as $n \to \infty$, so $ak_n - bh_n = 0$ for some n (since this quantity is an integer). So $x = r_n$, a contradiction (since r_{n+2k}, $k \geq 0$, is a monotone sequence converging to x).

Second, since $x = \lim_{n\to\infty} [a_0, \dots, a_n] = \lim_{n\to\infty} a_0 + \dfrac{1}{[a_1, \dots, a_n]} = a_0 + \dfrac{1}{[a_1, \dots]}$, we can, as before, recover a_0 from x as $a_0 = \lfloor x \rfloor$. This in turn, using the same proof as for finite continued fractions, yields

If $x = [a_0, a_1, \dots] = [b_0, b_1, \dots]$ with $a_i, b_i \in \mathbb{Z}$ and $a_i, b_i \geq 1$ for all $i \geq 1$, then $a_i = b_i$ for all $i \geq 0$.

Third, the convergents $r_n = \dfrac{h_n}{k_n}$ give better rational approximations than any other rational numbers we might choose:

If $x \notin \mathbb{Q}$ and $b \in \mathbb{Z}$ with $1 \leq b \leq k_n$, then for any $a \in \mathbb{Z}$, $|x - \dfrac{a}{b}| \geq |x - \dfrac{h_n}{k_n}|$. In fact, if $1 \leq b < k_{n+1}$, then $|bx - a| \geq |k_n x - h_n|$.

To see this, suppose not; suppose $1 \leq b < k_{n+1}$ and, for some a, $|bx - a| < |k_n x - h_n|$. We can assume that $(a, b) = 1$. We first solve the system of equations $h_n\alpha + h_{n+1}\beta = a$, $k_n\alpha + k_{n+1}\beta = b$; the solutions are $\alpha = (-1)^{n+1}(k_{n+1}a - h_{n+1}b), \beta = (-1)^{n+1}(h_n b - k_n a)$. Note that $\alpha, \beta \neq 0$, since otherwise $\dfrac{a}{b} = \dfrac{h_n}{k_n}$ or $\dfrac{h_{n+1}}{k_{n+1}}$, so $|bx - a| = |k_n x - h_n|$ or $b = k_{n+1}$. Also, if $\alpha < 0$, then $k_n\alpha + k_{n+1}\beta = b$ implies $k_{n+1}\beta = b - k_n\alpha > b > 0$, so $\beta > 0$. And if $\alpha > 0$, then $k_n\alpha + k_{n+1}\beta = b$ implies $k_{n+1}\beta = b - k_n\alpha < k_{n+1} - k_n\alpha < k_{n+1}$, so $\beta < 1$, so $\beta < 0$. So α and β have opposite signs. On the other hand, from before, we know that $xk_n - h_n$ and $xk_{n+1} - h_{n+1}$ have oposite signs, as well, so $\alpha(xk_n - h_n)$ and $\beta(xk_{n+1} - h_{n+1})$ have the <u>same</u> sign. Then: $xb - a = x(k_n\alpha + k_{n+1}\beta) - (h_n\alpha + h_{n+1}\beta) = \alpha(xk_n - h_n) + \beta(xk_{n+1} - h_{n+1})$, so $|xb - a| = |\alpha| \cdot |xk_n - h_n| + |\beta| \cdot |xk_{n+1} - h_{n+1}| \geq |\alpha| \cdot |xk_n - h_n| \geq |xk_n - h_n|$, a contradiction. So no such a exists!