Math 445 Number Theory

November 3, 2004

From last time: If $x \notin \mathbb{Q}$ and $b \in \mathbb{Z}$ with $1 \le b < k_{n+1}$, then for any $a \in \mathbb{Z}$, $|bx - a| \ge |k_n x - h_n|$.

Another sense in which convergents are the best possible rational approximations:

If $x \notin \mathbb{Q}$ and $a, b \in \mathbb{Z}$ have $|x - \frac{a}{b}| < \frac{1}{2b^2}$, then $\frac{a}{b} = \frac{h_n}{k_n}$ for some n.

The idea: if not, then $|ak_n - bh_n| \ge 1$ for every n. But since $k_n \to \infty$ as $n \to \infty$, there is an n with $k_n \le b < k_{n+1}$. Then from above we know that $|xk_n - h_n| \le |xb - a| = |x - \frac{a}{b}| \cdot |b| < \frac{1}{2b^2}|b| = \frac{1}{2b}$. So $|x - \frac{h_n}{k_n}| < \frac{1}{2bk_n}$, and then $\frac{1}{bk_n} \le \frac{|bh_n - ak_n|}{bk_n} = |\frac{a}{b} - \frac{h_n}{k_n}| = |(\frac{a}{b} - x) + (x - \frac{h_n}{k_n})| \le |\frac{a}{b} - x| + |x - \frac{h_n}{k_n}| < \frac{1}{2b^2} + \frac{1}{2bk_n}$. So $\frac{1}{2bk_n} = \frac{1}{bk_n} - \frac{1}{2bk_n} \le \frac{1}{2b^2}$, so $2b^2 < 2bk_n$, so $b < k_n$, a contradiction. So $\frac{a}{b} = \frac{h_n}{k_n}$ for some n.

Pell's Equation: solve $x^2 - ny^2 = N$ with $x, y \in \mathbb{Z}$. (WOLOG, $x, y \ge 0$)

If $n < 0$, then $N = x^2 - ny^2 \ge x^2 + y^2 \Rightarrow x, y \le \sqrt{N}$; can check all cases.

If $n = m^2$ is a perfect square, then $N = x^2 - ny^2 = (x - my)(x + my) \Rightarrow x - my = a, x + my = b$ with $ab = N$, and so $2x = a + b, 2my = b - a$. Again, we can just check all factorizations $N = ab$ to see what works.

If $n > 0$ is not a perfact square, then we can use the continued fraction expansion of $\sqrt{n}$ to shed light on the solutions. If $N > 0$, then $\quad N = x^2 - ny^2 = (x - \sqrt{n}y)(x + \sqrt{n}y)$, so $0 < \frac{N}{x + \sqrt{n}y} = x - \sqrt{n}y$, so $\frac{|N|}{|x + \sqrt{n}y| \cdot |y|} = |\sqrt{n} - \frac{x}{y}|$.

And since $x - \sqrt{n}y > 0$, $x > \sqrt{n}y$, so $\frac{x}{\sqrt{n}y} > 1$ so $\frac{x}{\sqrt{n}y} + 1 = \frac{x + \sqrt{n}y}{\sqrt{n}y} > 2$, so $x + \sqrt{n}y > 2\sqrt{n}y$ so

$$|\sqrt{n} - \frac{x}{y}| = \frac{|N|}{|x + \sqrt{n}y| \cdot |y|} < \frac{|N|}{2\sqrt{n}|y| \cdot |y|} = \frac{|N|}{\sqrt{n}} \cdot \frac{1}{2y^2} .$$

So if $0 < N < \sqrt{n}$, then $x^2 - ny^2 = N \Rightarrow |\sqrt{n} - \frac{x}{y}| < \frac{1}{2y^2} \Rightarrow \frac{x}{y}$ is a <u>convergent</u> of $\sqrt{n}$.

(A similar argument works for $-\sqrt{n} < N < 0$.)

Which makes it more interesting to understand the convergents of $\sqrt{n}$! The basic idea: x has a repeating continued fraction expansion $x = [a_0, \dots, a_n, \overline{b_0, \dots, b_m}] \Leftrightarrow x = r + s\sqrt{t}$ for some $r, s \in \mathbb{Q}$, $t \in \mathbb{Z}$.

To see this, set $\alpha = [\overline{b_0, \dots, b_m}]$, so $x = [a_0, \dots, a_n, \alpha]$. If $[a_0, \dots, a_n] = \frac{h_n}{k_n}$, then $x = [a_0, \dots, a_n, \alpha] = \frac{h_n\alpha + h_{n-1}}{k_n\alpha + k_{n-1}}$. <u>If</u> $\alpha = r_0 + s_0\sqrt{t}$, then $x = \frac{h_n(r_0 + s_0\sqrt{t}) + h_{n-1}}{k_n(r_0 + s_0\sqrt{t}) + k_{n-1}} = \frac{(h_n s_0)\sqrt{t}) + (h_n r_0 + h_{n-1})}{(k_n s_0)\sqrt{t}) + (k_n r_0 + h_{n-1})} = \frac{((h_n s_0)\sqrt{t}) + (h_n r_0 + h_{n-1}))(k_n s_0)\sqrt{t}) - (k_n r_0 + h_{n-1})}{k_n^2 s_0^2 t - (k_n r_0 + h_{n-1})^2}$

$$= \frac{h_n k_n s_0^2 t - (h_n r_0 + h_{n-1})(k_n r_0 + h_{n-1})}{k_n^2 s_0^2 t - (k_n r_0 + h_{n-1})^2} + \frac{((k_n s_0)(h_n r_0 + h_{n-1}) - (h_n s_0)(k_n r_0 + h_{n-1}))}{k_n^2 s_0^2 t - (k_n r_0 + h_{n-1})^2}\sqrt{t} = r + s\sqrt{t} \text{ with } r, s \in \mathbb{Q} .$$