# Math 445 Number Theory

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$x$ has a repeating continued fraction expansion $x = [a_0, \dots, a_n, \overline{b_0, \dots, b_m}] \Leftrightarrow x = r + s\sqrt{t}$ for some $r, s \in \mathbb{Q}$ , $t \in \mathbb{Z}$ . Last time: enough to show this for $\alpha = [\overline{b_0, \dots, b_m}] = [b_0, \dots, b_m, \alpha]$ . Then for $[b_0, \dots, b_m] = \dfrac{h'_m}{k'_m}$ , $\alpha = \dfrac{h'_m\alpha + h'_{m-1}}{k'_m\alpha + k'_{m-1}}$ , so $k'_m\alpha^2 + k'_{m-1}\alpha = h'_m\alpha + h'_{m-1}$ , so $\alpha$ is the solution of a quadratic equation with rational coefficients, so $\alpha = r_0 + s_0\sqrt{t}$ , as desired. The converse ($\Leftarrow$) direction follows an argument parallel to one of your homework questions; our further explorations will not need this direction.

In what follows, for $x = \dfrac{a + \sqrt{d}}{b}$ , it will be useful to have the notation $x' = \dfrac{a - \sqrt{d}}{b}$ for the *conjugate* of $x$, that is, the other root of the quadratic having $x$ as root. Our main result on periodic continued fractions is: If $x = \sqrt{n} + \lfloor\sqrt{n}\rfloor$, then $x = [\overline{a_0, \dots, a_k}]$ is *purely periodic.*

To see this, note that $x' = \lfloor\sqrt{n}\rfloor - \sqrt{n}$ , so $-1 < x' < 0$ . If we set $x = [a_0, \dots, a_i + x_i] = [a_0, \dots, a_i, \zeta_i]$ (so $\zeta_i = \dfrac{1}{x_i}$ and $a_{i+1} = \lfloor\zeta_i\rfloor$) then from our homework we know that (since $\sqrt{n} = [b_0, b_1, \dots] = [a_0 - \lfloor\sqrt{n}\rfloor, a_1, a_2, \dots]$) $x_i = \dfrac{\sqrt{n} - m_i}{q_i}$ and $\zeta_{i+1} = \dfrac{q_i}{\sqrt{n} - m_i} = \dfrac{\sqrt{n} + m_i}{q_{i+1}}$ .

So $x_{i+1} = \zeta_{i+1} - a_{i+1}$ , where $q_i q_{i+1} = n - m_i^2$ (which, inductively, defines $q_{i+1}$) , $a_{i+1} = \lfloor\zeta_{i+1}\rfloor$, so $\dfrac{\sqrt{n} + m_i}{q_{i+1}} = a_{i+1} + \dfrac{\sqrt{n} - m_{i+1}}{q_{i+1}}$ , and so $m_{i+1} = a_{i+1}q_{i+1} - m_i$ (which, inductively, defines $m_{i+1}$) . In other words, the formulas $q_{i+1} = \dfrac{n - m_i^2}{q_i}$ , $a_{i+1} = \lfloor\dfrac{\sqrt{n} + m_i}{q_{i+1}}\rfloor$ , and $m_{i+1} = a_{i+1}q_{i+1} - m_i$ allow us to inductively define each of these symbols, starting from $m_0 = \lfloor\sqrt{n}\rfloor$ and $q_0 = 1$ .

The key to the proof is that $-1 < \zeta'_i < 0$ for all $i$; the proof may be found at the end of the day's notes. This implies that $\lfloor\dfrac{-1}{\zeta'_{i+1}}\rfloor = \lfloor a_i - \zeta'_i\rfloor = a_i$, since $a_i < a_i - \zeta'_i < a_i + 1$ . So $a_i$ can be recovered from $\zeta_{i+1}$ .

We know, from homework, that the continued fraction for $\sqrt{n}$ and therefore for $\sqrt{n} + \lfloor\sqrt{n}\rfloor$ (since they agree in all but the first term), becomes periodic; past a certain point $k$, there is an $m > 0$ with $a_{k+s+m} = a_{k+s}$ for all $s \geq 0$. That is, $\zeta_k = \zeta_{k+m}$ . Let $k$ and $m$ be the smallest such numbers (i.e., $k$ = place where periodicity starts, $m$=length of the shortesst period). We *claim:* $k = 0$ . But this is just because if $k > 0$, then $\zeta_k = \zeta_{k+m} \Rightarrow \zeta'_k = \zeta'_{k+m} \Rightarrow a_{k-1} = \lfloor\dfrac{-1}{\zeta'_k}\rfloor = \lfloor\dfrac{-1}{\zeta'_{k+m}}\rfloor = a_{k+m-1} \Rightarrow \dfrac{1}{\zeta_{k-1} - a_{k-1}} = \zeta_k = \zeta_{k+m} = \dfrac{1}{\zeta_{k+m-1} - a_{k+m-1}} = \dfrac{1}{\zeta_{k+m-1} - a_{k-1}} \Rightarrow \zeta_{k-1} = \zeta_{(k-1)+m}$ , contradicting our choice of $k$ .

So $k = 0$ ; and so there is an $m > 0$ so that $a_{m+s} = a_s$ for all $s \geq 0$ . So $\sqrt{n} + \lfloor\sqrt{n}\rfloor = [\overline{a_0, \dots, a_{m-1}}] = [a_0, \overline{a_1, \dots, a_{m-1}, a_0}]$ . Note that $a_0 = 2\lfloor\sqrt{n}\rfloor$, so $\sqrt{n} = [\lfloor\sqrt{n}\rfloor, \overline{a_1, \dots, a_{m-1}, 2\lfloor\sqrt{n}\rfloor}]$.

Now back to Pell's Equation! We know that if $|N| < \sqrt{n}$, then every solution to $x^2 - ny^2 = N$ has $\dfrac{x}{y}$ = a convergent of $\sqrt{n}$. But as we have just seen, $\sqrt{n} + \lfloor\sqrt{n}\rfloor = [\overline{2\lfloor\sqrt{n}\rfloor, a_1, \dots, a_{m-1}}]$ , and this will allow us to shed light on $h_i^2 - nk_i^2$, to understand Pell's equation better.

$\sqrt{n} + \lfloor\sqrt{n}\rfloor = [\overline{2\lfloor\sqrt{n}\rfloor, a_1, \dots, a_{m-1}}]$ means (with $a_0 = \lfloor\sqrt{n}\rfloor$) that $\sqrt{n} = [a_0, \overline{a_1, \dots, a_{m-1}, 2a_0}]$

Wherever we choose to stop the continued fraction expansion of $\sqrt{n}$, $\sqrt{n} = [a_0, \dots, a_s, \zeta_{s+1}] = [a_0, \dots, a_s, \dfrac{\sqrt{n} + m_s}{q_{s+1}}]$, we find that

$\sqrt{n} = \dfrac{\frac{\sqrt{n}+m_s}{q_{s+1}}h_s + h_{s-1}}{\frac{\sqrt{n}+m_s}{q_{s+1}}k_s + k_{s-1}} = \dfrac{(\sqrt{n} + m_s)h_s + q_{s+1}h_{s-1}}{(\sqrt{n} + m_s)k_s + q_{s+1}k_{s-1}}$ . Using this, we can calculate what $h_s^2 - nk_s^2$ equals; we will do this next time.

Proof of $-1 < \zeta_1' < 0$: Note that $\zeta_i = \dfrac{\sqrt{n} + m_{i-1}}{q_i}$ , so

$\zeta_{i+1} = \dfrac{1}{\zeta_i - a_i} = \dfrac{1}{\frac{\sqrt{n}+m_{i-1}}{q_i} - a_i} = \dfrac{q_i}{\sqrt{n} + m_{i-1} - a_i q_i} = \dfrac{q_i\sqrt{n} - (m_{i-1} - a_i q_{i+1})q_i}{n - (m_{i-1} - a_i q_i)^2}$ . Then

$\zeta_i' = \dfrac{-\sqrt{n} + m_{i-1}}{q_i}$, and

$\dfrac{1}{\zeta_i' - a_i} = \dfrac{1}{\frac{-\sqrt{n}+m_{i-1}}{q_i} - a_i} = \dfrac{q_i}{(m_{i-1} - a_i q_i) - \sqrt{n}} = \dfrac{q_i((m_{i-1} - a_i q_i) + \sqrt{n}}{(m_{i-1} - a_i q_i)^2 - n} =$

$\dfrac{-q_i\sqrt{n} - (m_{i-1} - a_i q_{i+1})q_i}{n - (m_{i-1} - a_i q_i)^2} = \zeta_{i+1}'$ .

But $x = \zeta_0$, so $-1 < \zeta_0' < 0$ ; then we have, by induction, $-1 < \zeta_i' \Rightarrow \zeta_i' - a_i < -1 \Rightarrow -1 < \dfrac{1}{\zeta_i' - a_i} =$ $\zeta_{i+1}' < 0$ .