Math 445 Number Theory

November 17, 2004

Theorem: If abc is square-free, then $ax^2+by^2+cz^2=0$ has a (non-trfvial!) solution $x,y,z\in\mathbb{Z} \Leftrightarrow$ a,b,c do not all have the same sign, and each of the equations
$w^2\equiv -ab \pmod{c}, w^2\equiv -ac \pmod{b}, w^2\equiv -bc \pmod{a}$ have solutions.

($\Leftarrow$:) After possible renaming variables and taking negatives, we may assume that $a>0$ and $b,c<0$. Suppose $r^2\equiv -ab \pmod{c}$ and $aA\equiv 1 \pmod{c}$. Then for any $x,y\in\mathbb{Z}$, mod c we have $ax^2+by^2+cz^2\equiv ax^2+by^2\equiv aA(ax^2+by^2)\equiv A(a^2x^2+aby^2)=A(a^2x^2-r^2y^2)=A(ax-ry)(ax+ry)\equiv(x-Ary+0z)(ax+ry+0z)$. Similarly, mod b (with $s^2\equiv -ac$) we have $ax^2+by^2+cz^2\equiv(x+0y-Asz)(ax+0y+sz)$ and, mod a (with $t^2\equiv -bc$ and $bB\equiv 1$) we have $ax^2+by^2+cz^2\equiv(0x+y-Btz)(0x+by+tz)$.
Using the Chinese Remainder Theorem, we can solve $\alpha\underset{a}{\equiv}1,\alpha\underset{b}{\equiv}1,\alpha\underset{c}{\equiv}0$, $\beta\underset{a}{\equiv}-A,\beta\underset{b}{\equiv}0,\beta\underset{c}{\equiv}1$, $\gamma\underset{a}{\equiv}0,\gamma\underset{b}{\equiv}-As,\gamma\underset{c}{\equiv}-Bt$, $\delta\underset{a}{\equiv}a,\delta\underset{b}{\equiv}a,\delta\underset{c}{\equiv}0$, $\epsilon\underset{a}{\equiv}r,\epsilon\underset{b}{\equiv}0,\epsilon\underset{c}{\equiv}b$, and $\eta\underset{a}{\equiv}0,\eta\underset{b}{\equiv}s,\eta\underset{c}{\equiv}t$. Then, mod abc , $ax^2+by^2+cz^2\equiv(\alpha x+\beta y+\gamma z)(\delta x+\epsilon y+\eta z)$. Then we need a

Lemma : If $\lambda,\mu,\nu\in\mathbb{R}$ and positive, with $\lambda\mu\nu=M\in\mathbb{Z}$, then for any $\alpha,\beta,\gamma\in\mathbb{Z}$, $\alpha x+\beta y+\gamma z\equiv 0 \pmod{M}$ has a solution with $x,y,z\in\mathbb{Z}$, $(x,y,z)\neq(0,0,0)$, and $|x|\leq\lfloor\lambda\rfloor,|y|\leq\lfloor\mu\rfloor,|z|\leq\lfloor\nu\rfloor$.

The proof is simply that, for $0\leq x\leq\lfloor\lambda\rfloor,0\leq y\leq\lfloor\mu\rfloor,0\leq z\leq\lfloor\nu\rfloor$, we have $(1+\lfloor\lambda\rfloor)(1+\lfloor\mu\rfloor)(1+\lfloor\nu\rfloor)>\lambda\mu\nu=M$ triples (x,y,z) , and so $\alpha x+\beta y+\gamma z\equiv\alpha x_1+\beta y_1+\gamma z_1$ for some pair of triples, and so $\alpha(x-x_1)+\beta(y-y_1)+\gamma(z-z_1)\equiv 0$.

Setting $\lambda=\sqrt{bc},\mu=\sqrt{-ac},\nu=\sqrt{-ab}$, we then can solve $\alpha x+\beta y+\gamma z\equiv 0 \pmod{abc}$ (so $ax^2+by^2+cz^2\equiv 0 \pmod{abc}$) with $|x|\leq\sqrt{bc},|y|\leq\sqrt{-ac},|z|\leq\sqrt{-ab}$. But since abc is square-free, none of these square roots are integers (unless they are 1). So $x^2\leq bc,y^2\leq -ac,z^2\leq -bc$, and equality occurs if any only if the corresponding right-hand side is 1.

Then, unless $b=c=-1$, we have $x^2<bc$ and $abc|ax^2+by^2+cz^2$ with $ax^2+by^2+cz^2\leq ax^2<abc$ and $ax^2+by^2+cz^2\geq by^2+cz^2>b(-ac)+c(-ab)=-2abc$. [The last inequality is reversed, since $b,c<0$. It is strict, unless $a=1$ as well.] So $ax^2+by^2+cz^2=0$ or $=-abc$. In the first case we are done; in the second, setting $X=-by+xz,Y=ax+yz,Z=z^2+ab$ we have $aX^2+bY^2+cZ^2=a(-by+xz)^2+b(ax+yz)^2+c(z^2+ab)^2=(ab^2y^2-2abxyz+ax^2z^2)+(a^2bx^2+2abxyz+by^2z^2)+(cz^4+2abcz^2+a^2b^2c)=(ax^2+by^2+cz^2)z^2+ab^2y^2+a^2bx^2+2abcz^2+a^2b^2c=-abcz^2+ab^2y^2+2abcz^2+a^2bx^2+a^2b^2c=ab(ax^2+by^2+cz^2)+a^2b^2c=(ab)(-abc)+(ab)(abc)=0$. This gives a non-trivial solution, unless $0=-by+xz,0=ax+yz,0=z^2+ab$, so $z^2=-ab$, so $a=1,b=-1$ since ab is square-free; and then $(x,y,z)=(1,1,0)$ is a solution.

Finally, in the special case $b=c=-1$, we have $w^2\equiv -bc=-1 \pmod{a}$, has a solution, so every prime factor p of a also has $w^2\equiv -1 \pmod{p}$, so $p\equiv -1 \pmod{4}$ for every prime factor, so $y^2+z^2=a$ has a solution, so $(1,y,z)$ is a solution to $ax^2+by^2+cy^2=ax^2-y^2-z^2=0$, as desired.