

## Math 445 Number Theory

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A more powerful (read: faster) factoring algorithm: the Quadratic Sieve.

The starting point for the quadratic sieve is the observation that if  $n = ab$  is an odd composite number, then  $n$  is a difference of squares  $n = ab = (\frac{a+b}{2})^2 - (\frac{a-b}{2})^2$ . Fermat used this to describe a factoring algorithm:

Starting with  $a = \lfloor \sqrt{n} \rfloor$ , compute  $(a+k)^2 - n = a_k$  for  $k \in \mathbb{N}$  and look for a  $k$  with  $a_k = b^2$  a perfect square, then  $a+k-b|n$  and we have (probably) found a proper factor. This approach is fast if  $n$  has a factor close to  $\sqrt{n}$ , but in general (since a random number is more likely to have a small factor than one close to its square root) this algorithm is slower than trial division.

Fermat's idea was improved in the 1920's, by Kraitchik, whose instead proposed to find an  $a$  so that  $a^2 \equiv b^2 \pmod{n}$  for some  $b$ , i.e.,  $n|a^2 - b^2 = (a-b)(a+b)$ ; the gcd of  $n$  with  $a-b$  or  $a+b$  is then likely to produce a proper factor. Kraitchik's idea was to start as with fermat, to compute  $(a+k)^2 - n = a_k$ , but instead to find a collection of  $k$ 's,

$$k_1, \dots, k_r$$

so that  $a_{k_1} \cdots a_{k_r} = b^2$  was a perfect square. Then  $(a+k_1) \cdots (a+k_r) \equiv b^2 \pmod{n}$ , and we may have found a proper factor of  $n$ . And the approach to finding the right  $a_i$  was to search for  $i$  so that  $(a+i)^2 - n = a_i$  is a product of "small" primes. The idea is that with enough number, all of which are products of the same collection of small primes, we can eventually find some subset of them whose product is a square. A square has a prime factorization with all even exponents; while the exponent on each prime in the factorization of  $a_{k_1} \cdots a_{k_r}$  is the sum of the exponents in the factorization of each of the  $a_i$ . So we look for  $a_i$ 's so that the sum of the exponents for each prime, summed over the  $i$ , is always even. The algorithm proceeds by choosing a bound  $B$  on the primes in a factorization we are willing to keep; the set of primes  $p$  less than or equal to  $B$  is called the *factor base* of the algorithm. We keep all  $a_k$  whose prime factorizations involve only primes in the factor base, and continue to collect  $a_i$  until we can find a subset of them whose product is a square. How do we find such a subset? Linear algebra!